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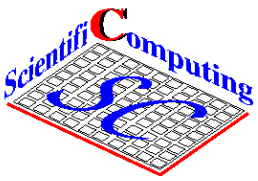

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Abstract:

Heterogeneous materials at the micro-structural level are usually subjected to several uncertainties. These materials behave according to an elastoplastic model, but with uncertain parameters. The present review discusses recent developments in numerical approaches to these kinds of uncertainties, which are modelled as random fields like Young's modulus, yield stress etc.

To give full description of random phenomena of elastoplastic materials one needs adequate mathematical framework. The probability theory and theory of random fields fully cover that need. Therefore, they are together with the theory of stochastic finite element approach a subject of this review.

The whole group of different numerical stochastic methods for the elastoplastic problem has roots in the classical theory of these materials. Therefore, we give here the classical formulation of plasticity in very concise form as well as some of often used methods for solving this kind of problems.

The main issues of stochastic elastoplasticity as well as stochastic problems in general are stochastic partial differential equations. In order to solve them we must discretise them. Methods of solving and discretisation are called stochastic methods. These methods like Monte Carlo, Perturbation method, Neumann series method, stochastic Galerkin method as well as some other very known methods are reviewed and discussed here.

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Contents

Chapter 1

Introduction

1.1 Overview

Stochastic systems have recently become fast growing area of scientific research. The increased interest in their simulation comes from the fact that systems are always subjected to random external influences or are themselves uncertain or very often both of these.

In general the uncertainties are classified as epistemic or aleatory uncertainties. Aleatory uncertainties are associated with the inherent variables of nature (e.g. due to heterogeneities in materials). Therefore, their influence cannot be reduced, but there is a high developed mathematical theory dealing with this kind of problem. However, epistemic uncertainties arise due to our lack of knowledge. We can reduce their influence by collecting data, although there is no such high developed mathematics dealing with this kind of problem. Sometimes epistemic uncertainties are traded-off for aleatory uncertainties for ease of propagating them through the governing equations using advanced mathematical tools.

The strategy of describing the uncertainties [120] consist of several common used methods [119]:

- Probabilistic or stochastic models, where uncertainties are modelled as random variables or random fields depending on whether they are specialised to a fixed location or a function of location in their continuum. If there is an uncertain time dependent function then it is modelled as stochastic process [2, 26, 64, 62, 97, 104, 175].
- Worst-case scenarios, trying to provide bounds using interval analysis, convex models etc.,e.g. [125].
- Fuzzy set theory, where parameters are described as possibility functions, describing their degree of belonging to a set e.g. [45].
- Evidence theory, creating upper and lower bounds on the likelihood of events e.g. [152].

Probabilistic or stochastic methods give us a mathematically most detailed description. They require information on the statistics of system properties. This can be disadvantage since these statistics are very hard to obtain. If the physical system is described by a partial differential equation then this method leads to a stochastic differential equation. Stochastic differential equations can be broadly classified into two categories: stochastic differential equations (SDE) with the random forcing and SDE with the random coefficient [89, 88]. Sometimes we have the combination of both, the random forcing and the random coefficient. The uncertainty associated with the coefficient term is generally attributed to the sampling/testing error in estimating the material properties or especially in geomechanics, due to inherent variability of the soil. The uncertainty in the forcing term arises when the material is subjected to dynamic loads like wind, wave or earthquake load.

The main concentration of this review is a stochastic approach to the classical theory of elastoplasticity [164]. In other words, we use assumption that heterogeneous material behaves according to an elastoplastic model. Heterogeneities at the micro-structural level are usually subjected to a number of uncertainties, so that elastoplastic model is not any more deterministic but stochastic with uncertain parameters, which are modelled as random fields like Young's modulus, yield stress etc. Therefore, the evolution law for the internal variables becomes a stochastic law. For solving this kind of equation recently has been used stochastic finite element method. The main aim of this method is to provide an efficient alternative to the time-costly Monte-Carlo simulation [156]. First attempt to deal with uncertainties through elastoplastic constitutive equations was work of M. Anders and M. Hori through the perturbation method [3, 4]. Recently, Jeremic [80, 160, 161] dealt with this kind of problem using the Fokker-Plank equation approach based on the work of Kavvas [87], who obtained a generic Eulerian-Lagrangian form of the Fokker-Plank equation, exact to second-order, corresponding to any nonlinear ordinary differential equation with random forcing and random coefficient. There are several approaches in formulating stochastic finite element method (SFEM) [3, 4]:

- perturbation method [34, 71, 93, 162], which is applicable to the general nonlinear problems, but it is restricted to cases when fluctuations of the uncertainties are small. This method is based on perturbation expansion at the stochastic mean behaviour using just first term of the expansion.
- multiplicative Neumann expansion of the governing stochastic operator [24, 190].
- approach based on orthogonal expansion in Hilbert space, globally opposite to the local Taylor expansion.

Theory of stochastic elastoplastic problem is based on the theory of Ghanem and Spanos [53, 57, 59], Anders and Hori [3, 4], Jeremic [80, 160, 161] and the work of Matthies and Kesse [88, 89, 119, 120, 121, 122]).

This report gives an overview of the mathematical foundation of stochastic fields and variables, then basic foundations of the classical theory of elastoplasticity and overview of the stochastic approach to that problem, giving also the basic knowledge of the numerical methods which can be used.

1.2 Terms and Symbols

A consistent notation is used through the text for all reviewed publications.

1.2.1 Glossary

The following abbreviations are used:

FE	Finite Element
FEM	Finite Element Method
FP	Fokker-Plank (equation)
FPKE	Fokker-Plank-Kolmogorov Equation
KL	Karhunen-Loeve (expansion, series)
NMOL	Numerical Method Of Lines
MC	Monte Carlo method
ODE	Ordinary Differential Equation
PCE	Polynomial Chaos Expansion
PDE	Partial Differential Equation
PDF	Probability Density Function
RF	Random Field
RV	Random Variable
SFEM	Stochastic Finite Element Method
SPDE	Stochastic Partial Differential Equation
SSFEM	Spectral Stochastic Finite Element Method
WCE	Wiener Chaos Expansion

1.2.2 Notation and Conventions

The following conventions are used:

$\boldsymbol{u}$	Vectors are small letters in a bold italic font
$\mathbf{u}$	Block vectors are small letters in a bold upright font
$\boldsymbol{K}$	Matrices are capital letters in a bold italic font
$\mathbf{K}$	Block matrices are capital letters in a bold upright font
γ, κ, ξ	Random variables and random fields are in Greek letters
$\boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\xi}$	Random vectors are in Greek letters
α, β, γ, i	These Greek letters are used for multi-indices
$f^{(\alpha)}$	A superscript multi-index in round brackets denotes the coefficient of a random variable in its PC expansion.

$f^m_{\leq k}$	The projection of the random variable f on the m-dimensional polynomial chaos of degree k.

1.2.3 Symbols

The following symbols are used throughout the text:

$\|\cdot\|_p$	Standard L^p-norm
$\|\cdot\|_\infty$	Standard L^∞-norm, essential supremum
$\|\cdot\|_{\rho,r}$	Hida distribution and test function norms
$\|\cdot\|_\rho$	Kondratev distribution and test function norms
$<\cdot,\cdot>$	Duality pairing
$(\cdot,\cdot)$	Scalar product
$\|\alpha\|$	Modulus of a multi-index α, defined as $\|\alpha\| = \sum \alpha_i, i \in \mathbb{N}$
α	Factorial of multi-index α, defined as $\alpha = \prod(\alpha_i), i \in \mathbb{N}$
$\odot$	The Wick-product
A	The drift vector in Fokker-Planck equation
α	multi-index
$\mathcal{B}$	Probability space, σ algebra of events
B	Diffusion coefficient in Fokker-Planck equation
$\mathbf{C}$	The forth-order elasticity tensor (constitutive tensor)
$\mathbf{C}^{-1}$	The forth-order compliance tensor
$\mathbf{C}^e$	The forth-order elasticity tensor in elastoplastic problem (constitutive tensor)
$\mathbf{C}^{ep}$	The elastoplastic forth-order constitutive tensor
$C_c^\infty(R)$	Space of infinitely often differentiable functions with compact support in $R \subset \mathbb{R}^d$.
$c_x(r)$	The covariance function
$\mathrm{cov}(\kappa_1, \kappa_2)$	Covariance of the random variables κ_1, κ_2 defined as $\mathrm{cov}(\kappa_1, \kappa_2) = \mathbf{E}((\kappa_1 - \mu_{\kappa_1})(\kappa_2 - \mu_{\kappa_2}))$
cov_γ	Covariance of Gaussian random field
$\boldsymbol{C_\kappa}$	The covariance matrix of a random vector $\boldsymbol{\kappa(\omega)}$.
d	Dimension of space; the spatial domain of the SPDE is $R \subset \mathbb{R}^d$
$\mathcal{D}$	Dissipation function
D	Fourth order tensor in general von Mises model
D^α	Partial derivative with respect to the multi-index
δ	Dirac delta function
∇	The Laplace operator
$\mathrm{erf}(x)$	The distribution function of a standard Gaussian random variable, $\mathrm{erf}(x) = F_{N(0,1)}(x)$
E	Young's modulus
$\mathbf{E}(\cdot)$	The expectation operator, $\mathbf{E}(g(\kappa)) = \int_\Omega g(\kappa(\omega))dP(\omega) = \int_{\mathbb{R}} g(\kappa)dF_\kappa(\kappa)$

$\mathbf{E}_p$	Generalised plastic strain $\mathbf{E}_p = (\boldsymbol{\varepsilon}_p, \boldsymbol{\vartheta})$
$\mathcal{E}_p$	Space of plastic strains
$\boldsymbol{\varepsilon}$	The total strain
$\boldsymbol{\varepsilon}_e$	The elastic part of the strain
$\boldsymbol{\varepsilon}_p$	The plastic part of the strain
$\boldsymbol{\varepsilon}^{\mathrm{dev}}$	Deviatoric part of total strain
$\boldsymbol{\varepsilon}^{\mathrm{sph}}$	Spherical part of total strain
$\boldsymbol{\varepsilon}_p^{\mathrm{dev}}$	Deviatoric part of plastic strain
$\boldsymbol{\varepsilon}_p^{\mathrm{sph}}$	Spherical part of plastic strain
$\boldsymbol{\varepsilon}_e^{\mathrm{dev}}$	Deviatoric part of elastic strain
$\boldsymbol{\varepsilon}_e^{\mathrm{sph}}$	Spherical part of elastic strain
$\dot{\boldsymbol{\varepsilon}}_p$	The rate of change of plastic strain
$\dot{\boldsymbol{\varepsilon}}_p^{\mathrm{dev}}$	The rate of change of deviatoric part of plastic strain
$\boldsymbol{\eta}_{ij}$	A random operator tensor representing the velocity of the point in the 9-dimensional stress space
$f_\gamma(x)$	The probability distribution function of a Gaussian random variable
$\boldsymbol{f}$	The vector of forces
$\boldsymbol{F}_i^{ext}$	External force in ith load step
$\boldsymbol{F}_i^{int}$	Internal force in ith load step
$\mathcal{F}$	The non-separable dual space of forces
$F_\kappa(k)$	Distribution function of a real-valued random variable κ, defined as $F_\kappa(k) = P_k(-\infty, k) = P(\kappa < k)$
$F_{\kappa(x),\kappa(y)}$	Joint probability distribution function
$\phi(\boldsymbol{\sigma}, \boldsymbol{\chi})$	Yield function
$\bar{\phi}(\boldsymbol{\sigma})$	Projection of yield function on the stress space when $\boldsymbol{\chi}$ is fixed
$\phi(x, \gamma)$	Nonlinear transformation used to transform a Gaussian field into a non-Gaussian one,$\kappa(x, \omega) = \phi(x, \gamma(x, \omega))$
$\varphi(x)$	Denotes real function where $x \in \mathbb{R}^d$
$g(\kappa)$	Random variable as a function of random variable
$g(\boldsymbol{\kappa})$	Function of a vector of random variables
G	Shear modulus
G_h	Gaussian space
$\mathbf{G}$	Matrix describing geometrical anisotropy
$\mathcal{G}$	Lipschitz domain
γ	Plastic multiplier
$\gamma(\omega)$	Gaussian random variable
$\gamma(x, \omega)$	Gaussian random field
$\bar{\gamma}$	Mean value of Gaussian random field
Γ_m	Gaussian probability measure in m dimensions, $d\Gamma_m(x) = (2\pi)^{-m/2} \exp(-\lvert x\rvert^2/2)dx$
Γ_p	Polynomial chaos of order p
$h_i(x)$	Univariate Hermite-polynomial of degree i for $x \in \mathbb{R}$

$H_\alpha(\omega)$	Multivariate Hermite-polynomial, indexed by the multi-index α. Defined as $H_\alpha(\omega) = \prod_{i\in\mathbb{N}} h_{\alpha_i}(\omega_i)$.
$H_0^1(R)$	Sobolev Hilbert space of once differentiable functions, completion of $C_c^\infty(R)$.
$\mathcal{H}$	Hilbert space, the spatial part of the solution
$\mathcal{H}_i$	Unnormalised Hermite polynomial
$\mathbf{H}$	The hardening modulus
$\mathbf{H}_{\text{kin}}$	The kinematic hardening modulus
$\mathbf{H}_{\text{iso}}$	The isotropic hardening modulus
$\mathcal{H}_{=p}$	Homogeneous chaos of degree p
$\mathcal{H}_{=p}^m$	m-dimensional homogeneous chaos of degree p
$\mathcal{H}_{\leq p}$	Polynomial chaos of degree p
$\mathcal{H}_{\leq p}^m$	m-dimensional polynomial chaos of degree p
$H(\kappa)$	Entropy of a random variable
$I_k(\boldsymbol{\sigma})$	kth stress invariant
K	Bulk modulus
$\mathcal{K}$	Set of admissible generalised stresses
int $\mathcal{K}$	Interior of admissible generalised stresses set
$\partial\mathcal{K}$	Yield surface (boundary of a set $\mathcal{K}$)
$\mathbf{K}$	Stiffness matrix
K_1	Modified Bassel function of the second kind of order 1
κ	A random field $\kappa(x,\omega), x \in R, \omega \in \Omega$
$\boldsymbol{\kappa}$	Vector of random variables. If $\kappa(x,\omega)$ is a random field, then $\boldsymbol{\kappa} = (\kappa_1, ..., \kappa_m)$ is the vector of random variables in its truncated KL-expansion.
κ_i	The uncorrelated random variables $\kappa_i(\omega)$ occuring in the KL expansion of a random field $\kappa(x,\omega)$
κ_l	Lognormal random variable
$\kappa^{(\alpha)}$	Projection of a random field onto the polynomial chaos, $\kappa^{(\alpha)}(x) = \mathbf{E}(\kappa(x,\cdot)H_\alpha)$
ξ	uncorrelated random variable
Ξ	The non-separable dual space of conjugate forces
$\boldsymbol{\chi}$	Internal forces
λ	Increment of the plastic consistency parameter
$\lambda_1, \lambda_2, ..$	The eigenvalues in the KL-expansion
$L^p(V)$	The standard Banach space (for the $0 < p < \infty$) or Hilbert space (for the $p = 2$) of functions $\in V$ whose p-th exponent is finitely integrable. $L^\infty(V)$ is the space of essentially bounded functions
$\mathbf{M}$	Gram matrix
μ_κ	The mean $\mu_\kappa = \mathbf{E}(\boldsymbol{\kappa})$ of the random variable $\boldsymbol{\kappa}$
$\boldsymbol{\mu}_{\boldsymbol{\kappa}}$	The mean vector of the random vector $\boldsymbol{\kappa}$
μ_l	The mean value of lognormal random variable

m	Denotes the number of independent random variables
ν	Poisson ratio
$\mathcal{N}(\mu, \sigma^2)$	A Gaussian random variable with mean μ and variance σ^2
$\mathcal{N}(0, 1)$	The standard Gaussian random variable
$(\Omega, \mathcal{B}, P)$	A probability space Ω:set of elementary events;$\mathcal{B}$:the σ algebra of events;P- probability measure
ω	an elementary event, $\omega \in \Omega$ or sequence of independent random variables $\omega = (\omega_1, \omega_2, ..)$
ω_i	$\omega_1, \omega_2, ..$ denote independent random variables
$\boldsymbol{\omega}$	Vector of m independent random variables $\boldsymbol{\omega} = (\omega_1, ..., \omega_m)^t$
Ω	Probability space, set of elementary events
p	Pressure
$p_\kappa(k)$	Probability density function of a random variable κ, defined as $p_\kappa(k) = dF_\kappa(k)/dk$
P	Probability measure
$P(\sigma_{ij}, t)$	Evolutionary probability density of the state variable tensor σ_{ij}
P_κ	Probability measure that is induced by a random variable κ on its range
Ψ	Helmholtz free energy
Ψ_e, Ψ_p	Elastic and plastic part of Helmholtz free energy
Ψ_G	Gibbs free energy
Ψ_{G_e}, Ψ_{G_p}	Elastic and plastic part of Gibbs free energy
$Q^{(m)}$	Quadrature formula
$Q_Z(\psi)$	High dimensional integration in Z integration points, $Q_Z(\psi) \approx E(\psi(\omega))$
R	The spatial region on which a stochastic field or the SPDE is defined $R \subset \mathbb{R}^d$
ρ	The coefficient of the correlation between κ_1 and κ_2
$\rho(\sigma_{ij}, t)$	Density in the stress space
S	The space of stresses
$\boldsymbol{s}$	Relative stress equal to $\boldsymbol{\sigma}^{\mathrm{dev}} - \boldsymbol{\sigma}_b$
$\boldsymbol{s}(x)$	Basis in finite dimensional space
$s(t, \omega)$	Random process
$(\mathcal{S})$	Abstract space of the stochastic functions
$(\mathcal{S})^{\rho,r}$	Hida distribution and test function spaces
$(\mathcal{S})^{\rho}$	Kondratev distribution and test function spaces
$\mathcal{S}(\mathbb{R}^d)$	Space of rapidly decreasing functions
$\mathcal{S}(\mathbb{R}^d)'$	Space of tempered distributions
S_l^m	Smolyak formula of level l in m dimensions
Σ	The space of stresses
σ_κ	Standard deviation of the random variable κ, defined as $\sigma_k = \sqrt{\mathrm{var}_\kappa}$
σ_l	Standard deviation of lognormal random variable
σ_y	The flow stress
$\bar{\sigma}$	Equivalent stress

σ_i	Principal value of stress
$\boldsymbol{\sigma}_e$	Elastic stress
$\boldsymbol{\sigma}_b$	Back stress
$\boldsymbol{\sigma}^{\mathrm{dev}}$	Deviatoric stress
$\sigma_{y,0}$	Initial flow stress
$\mathcal{T}$	Space defined with $\Sigma \times \Xi$
T_κ	Operator of SPDE
θ	Standard Gaussian random variable
ϑ	Internal variable in elastoplastic problem
$\boldsymbol{u}$	The displacement
$\mathcal{U}$	non-reflexive Banach space of displacement
U_h	The group of unitary operators
$V^{\pm}$	Bounding bodies
var_κ	Variance of the random variable κ, defined as $\mathrm{var}_\kappa = \mathbf{E}(\kappa^2) - \mu_\kappa^2$
$v(h)$	Semivariogram defined as $v(h) = \mathrm{var}(\kappa(x) - \kappa(x+h))$
$\mathcal{V}$	Finite dimensional space
$\boldsymbol{w}$	primal problem variable $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \boldsymbol{\vartheta})$ or dual problem variable $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\chi})$
x	Point in the spatial domain, $x \in R \subset \mathbb{R}^d$
y	Point in the spatial domain, $y \in R \subset \mathbb{R}^d$
Z	Hilbert space defined as $Z = \mathcal{U} \times \mathcal{E}_p \times \Theta$

Chapter 2

Basics

In this chapter we introduce basics of probability theory [79, 104]. It is necessarily needed when we speak about the problem of stochastic plasticity. The main role of concise theory presented here is to remind reader on definitions of random variables as well as to bring fixed notation throughout the review.

2.1 Random Variable

The triplet $(\Omega,\mathcal{B}, P)$ is a probability space with probability measure P and a σ algebra of events $\mathcal{B}$. A random variable (RV) κ is defined as a measurable map: $\kappa(\omega) : \Omega \to \mathcal{V}$ into a vector space $\mathcal{V}$, which is often equal to $\mathbb{R}$ [2, 31, 104]. If $\mathcal{V} = \mathbb{R}^d$, then κ is a random vector.

In probability theory, every random variable may be attributed to a function defined on a state space $(\Omega,\mathcal{B}, P)$ equipped with a probability distribution that assigns a probability P to every subset (more precisely every measurable subset) of its state space in such a way that the probability axioms (Kolmogorov axioms) are satisfied. That is, probability distributions are probability measures defined over a state space instead of the sample space Ω (set of all possible outcomes).
Let κ be a random variable with values in $\mathbb{R}$ and probability measure P, then its probability distribution $F_\kappa(a)$ is defined on $\mathbb{R}$ with values in $[0, 1]$:

$$F_\kappa = P\{\kappa < a\} = \int_{-\infty}^{a} dP. \tag{2.1.1}$$

The probability distribution function can be represented with a probability density function in terms of integrals. The probability density function is defined as derivative of probability distribution function $p_\kappa(a) = dF_\kappa/da$ (strictly in those cases when derivative exists). It has property of non-negative Lebesgue-integrable function such that the

probability distribution function is given by integral:

$$F_\kappa = \int_{-\infty}^{a} p_\kappa(t)dt \tag{2.1.2}$$

for any a. This implies that total integral must be equal to one [97].

Theorem 2.1.1. *In order that the function F on $\mathbb{R}$, with values in [0,1], be the distribution function of the probability distribution P on $\mathbb{R}$ it is necessary and sufficient that $F_\kappa(a)$ be increasing, left - continuous and tend to +1 as $a \to \infty$.*

For the random vector, definitions are little bit changed [41, 97, 104, 133]. The vector $\boldsymbol{\kappa}$ is called random vector if it is a finite collection of n random variables defined on a common probability space. The probability measure of the random vector is different from the probability measure of the random variable, since the vector is the joint outcome of several different measurements. Hence, a "joint probability distribution" of the random vector $\boldsymbol{\kappa}$ must be defined. The formal definition of the joint probability distribution function $F_\kappa(\boldsymbol{\kappa})$ of the random vector $\boldsymbol{\kappa}$ with a continuous range space is:

$$F_{\kappa_1,\kappa_2,..,\kappa_m}(k_1, k_2, .., k_m) = \int_{-\infty}^{k_1} ... \int_{-\infty}^{k_m} dP. \tag{2.1.3}$$

In addition, the random vector is characterised with bivariate statistics, from which the most important is a covariance or the measure of how much two random variables vary together :

$$\mathrm{cov}(\kappa_1, \kappa_2) = \mathbf{E}((\kappa_1 - \mu_{\kappa_1})(\kappa_1 - \mu_{\kappa_2})). \tag{2.1.4}$$

If two variables tend to vary together (that is, when one of them is above its expected value, then the other variable tends to be above its expected value too), then the covariance between two variables will be positive. On the other hand, if one of them is above its expected value and the other variable tends to be below its expected value, then the covariance between the two variables will be negative. Random variables whose covariance is zero are called uncorrelated, while the covariance of two random variables associated with different random fields is a cross-covariance. The units of the covariance are those of κ_1 times κ_2. Therefore, we define new function called a correlation function using standard deviation $\sigma(\kappa)$:

$$\rho = \frac{\mathrm{cov}[\kappa_2, \kappa_2]}{\sigma(\kappa_1)\sigma(\kappa_2)}, \tag{2.1.5}$$

to get obviously dimensionless measure of linear dependence. Very often for function ρ is used name "coefficient of correlation between κ_1 and κ_2 " instead mentioned name.

More difficult case is a case of a function of the random variable. Functions of the random variables, like random variables itself, allow the derivation of probabilities of a new random variables through functional relationships. Suppose that $g(\kappa)$ is a function of real variable κ [70], then the expectation of suitable function $g(\kappa(\omega))$ will be given as:

$$\mathbf{E}(g(\kappa)) = \int_\Omega g(\kappa)dP. \tag{2.1.6}$$

In the same way one may define the function of the random vector $g(\kappa_1, ..., \kappa_m)$ and get multivariate statistics (or expectation) [89]:

$$\mathbf{E}(g(k_1, k_2, .., k_m)) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(k_1, k_2, .., k_m) dF_{\kappa_1, \kappa_2, .., \kappa_m}(k_1, k_2, .., k_m). \tag{2.1.7}$$

For mutually independent random variables $\kappa_1, ..\kappa_m$ previous integral becomes:

$$\mathbf{E}(g(k_1, k_2, .., k_m)) = \int_{\mathbb{R}} ... \int_{\mathbb{R}} g(k_1, k_2, .., k_m) dF_{\kappa_1} dF_{\kappa_2} .. dF_{\kappa_m} \tag{2.1.8}$$

due to Fubini's theorem [142].

2.1.1 Gaussian Random Variables

The random variable $\gamma = \mathcal{N}(\mu, \sigma)$ having probability density function

$$f_\gamma(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp(-\frac{(\mathrm{x} - \mu)^2}{2\sigma^2}), \tag{2.1.9}$$

where μ is the mean value and σ the standard deviation, is called Gaussian random variable. The Gaussian distribution is also known as normal distribution. The random variable is known as normalised Gaussian random variable when its mean value and standard deviation (μ, σ) are equal to $(0, 1)$ respectively. Then the probability distribution function is known as standard normal distribution.

Gaussian random variables have property of independence in the case when they are uncorrelated. Linear combination of these variables is again Gaussian [61]. Independence and uncorrelation are main reasons why these variables are very often used for "coordinate systems" [100, 99, 12, 142].

2.1.2 Gaussian Spaces

If $\kappa : \Omega \to \mathbb{R}^n$ is a random variable and $p \in [1, \infty)$ is a constant, we define the L^p norm of κ by [131] :

$$\|\kappa\|_p = \int_\Omega (|\kappa(\omega)|^p \, dP(\omega))^{1/p}, \tag{2.1.10}$$

where for $p = \infty$ norm become

$$\|\kappa\|_\infty = \sup\{|\kappa(\omega)| : \omega \in \Omega\}. \tag{2.1.11}$$

For a probability space $(\Omega, \mathcal{B}, P)$ a linear normed space of random variables κ, defined with

$$\mathbf{E}\,|\kappa\,|^p = \int_\Omega |\kappa(\omega)|^p \, dP(\omega) < \infty, \tag{2.1.12}$$

where $p \geq 0$, is an $L^p(\Omega, \mathcal{B}, P)$ space (or simply an L^p space). The corresponding norm (2.1.10) then is defined by the usual formula:

$$\|\kappa\|_p = \{\mathbf{E}\,|\kappa|^p\}^{1/p}, \tag{2.1.13}$$

and L^p space becomes complete linear normed space called Banach space. In special case when $p = 2$, L^p space becomes L^2 space. L^2 space equipped with the scalar product:

$$(\kappa_1, \kappa_2) = \mathbf{E}[\kappa_1, \kappa_2] = \int_\Omega \kappa_1(\omega)\kappa_2(\omega)dP(\omega), \tag{2.1.14}$$

where κ_1 and κ_2 are random variables, is called a Hilbert space (i.e. complete inner product space). In the case of Gaussian random variables one speaks about Gaussian spaces [76], defined due to the next definition:

Definition 2.1.2. *A Gaussian linear space is a real linear space of random variables, defined on some probability space* $(\Omega,\mathcal{B}, P)$*, such that each variable in the space is centred Gaussian. Obviously, a Gaussian linear space is linear subspace of* $L^2_{\mathbb{R}}(\Omega,\mathcal{B}, P)$ *and we use the norm and inner product of* L^2 *on it. A Gaussian Hilbert space is a Gaussian linear space which is complete, i.e. a closed subspace of* $L^2_{\mathbb{R}}(\Omega,\mathcal{B}, P)$ *consisting of centred Gaussian random variables.*

which is supplied with the next theorem:

Theorem 2.1.3. *If* $G_h \subset L^2_{\mathbb{R}}(\Omega,\mathcal{B}, P)$ *is a Gaussian linear space, then its closure* $\overline{G_h}$ *in* L^2 *is a Gaussian Hilbert space.*

This theorem gives definition of the most important space of random variables - space of Gaussian random variables, which are very often used in practice.

2.1.3 Lognormal Random Variable

In practice Gaussian random variables are very often clearly inappropriate to use because of its symmetry, its infinite domain and the rate of decay of the tail of its density function. Some of examples are ground water flow and crack propagation. That brings some other kinds of random variables into consideration, like lognormal, which are given as a function of Gaussian random variable [56, 177].

The lognormal random variable κ_l is obtained from Gaussian random variable by taking its exponential:

$$\kappa_l = \exp(\mu_g + \sigma_g\gamma) \tag{2.1.15}$$

where γ is a standard normal variable. Moments of the lognormal random variable (mean value μ_l and standard deviation σ_l) are related to Gaussian's mean value μ_γ and standard deviation σ_γ through the following simple relationships:

$$\sigma_\gamma = \sqrt{\ln(1 + (\sigma_l/\mu_l)^2)} \tag{2.1.16}$$

$$\mu_\gamma = \ln\mu_l - 0.5\sigma_\gamma^2. \tag{2.1.17}$$

The measure how much two random variables vary together is defined through covariance function:

$$\mathrm{cov}(\kappa_{l_1}, \kappa_{l_2}) = (\exp\{\mathrm{cov}(\gamma_1, \gamma_2)\sigma_{\gamma_1}\sigma_{\gamma_2}\} - 1)\exp\{\mu_{\gamma_1} + \mu_{\gamma_2} + 0.5(\sigma_{\gamma_1}^2 + \sigma_{\gamma_2}^2)\}. \tag{2.1.18}$$

Chapter 3

Random Fields

Random processes are an infinite collection of random variables where random variables are indexed on a discrete or continuous "index set". This index set always corresponds to time t or spatial location x. The term "random process" or "stochastic process" is mostly used if the index set is the time variable, while the term "random field" is commonly applied for index sets of spatial locations i.e. uncertainties in physical quantities varying in time or in space. The mathematical definitions of the stochastic process and random fields are often the same [47].

3.1 Definitions

A random field is denoted as an indexed family of random variables :

$$\{\kappa(\omega, x), (\omega \in \Omega, x \in R, R \subset \mathbb{R}^d)\}; \tag{3.1.1}$$

on a common probability space $(\Omega, \mathcal{B}, P)$. In other words the random field is a measurable mapping

$$\kappa : R \times \Omega \to \mathbb{R}. \tag{3.1.2}$$

There are two approaches [2, 26, 131] in definition of the random field: probabilistic and measure theoretic characterisation.

The probabilistic approach defines the random field as a set of random variables specified by finite dimensional distribution functions i.e.

$$\kappa(x) = \kappa(x, \cdot) : \Omega \to \mathbb{R} \tag{3.1.3}$$

indexed by $x \in R$. Therefore, the random field is defined by specification of all finite dimensional distribution functions [41]:

$$F_{x_1,x_2,..,x_n}(\widehat{x}_1, .., \widehat{x}_n) = P(\kappa(x_1) \leq \widehat{x}_1, .., \kappa(x_n) \leq \widehat{x}_n) \tag{3.1.4}$$

with $\{x_1, x_2, ..., x_n\} \in R$ and $\{\widehat{x}_1, ..., \widehat{x}_n\} \in \mathbb{R}$ [41].

The measure theoretic approach [131] defines random field as a random variable that has as values functions on a region $R \subset \mathbb{R}^d$ i.e.

$$\kappa(\cdot, \omega) : R \to \mathbb{R}, \tag{3.1.5}$$

so that random field can be observed as realizations defined in subset $\Omega \subset \{\omega | \omega : R \to \mathbb{R}\}$.

The random field can be considered as a function of both elementary events $\omega \in \Omega$ and spatial positions $x \in \mathbb{R}^n$ [145]. According to Kolmogorov, necessary and sufficient conditions for existence of the random field are symmetry and consistency conditions represented by distribution functions:

$$F_{\kappa_1,\kappa_2,..\kappa_m}(k_1, k_2, ..k_m) = F_{\kappa_{i_1},\kappa_{i_2},..\kappa_{i_m}}(k_1, k_2, ..k_m) \tag{3.1.6}$$

$$F_{\kappa_1,\kappa_2,..\kappa_m,\kappa_{m+1},..,\kappa_{m+k}}(k_1, k_2, ..k_m) = F_{\kappa_{i_1},\kappa_{i_2},..\kappa_{i_m}}(k_1, k_2, ..k_m). \tag{3.1.7}$$

3.2 Generalised Random Fields

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of all infinitely differentiable real functions $\varphi(x)$, $x \in \mathbb{R}^d$ on a d-dimensional Euclidean space $\mathbb{R}^d$ [39], decreasing at infinity, together with all their derivatives, more rapidly than any negative power $|x|^{-k}$, $k = \{1, 2, ...\}$.

The topology on $\mathcal{S}(\mathbb{R}^d)$ is given by a countable set of norms:

$$\|\varphi\|_j = \sup\{\max(1, |x|^j) \sum_{|q|<j} |\varphi^q(x)|\} \tag{3.2.1}$$

where $j = \{0, 1, ..\}$ and $q = \{q_1, .., q_d\}$ is the set of non-negative integers, such that:

$$|q| = q_1 + ... + q_d \tag{3.2.2}$$

and

$$\varphi^q(x) = \frac{\partial^{|q|}\varphi(x)}{\partial^{q_d} x_d .. \partial^{q_1} x_1}, \tag{3.2.3}$$

where $x = (x_1, ..., x_d) \in \mathbb{R}^d$ (with an obvious modification when some $q_i = 0$, $i = \{1, ..., d\}$). In this topology $\mathcal{S}(\mathbb{R}^d)$ is a locally convex nuclear space.

Next, we denote by $\mathcal{S}' = \mathcal{S}(\mathbb{R}^d)'$ the space of all real linear continuous functionals on $\mathcal{S}(\mathbb{R}^d)$ equipped with the weak topology . The elements of the space $\mathcal{S}(\mathbb{R}^d)'$ are said to be real tempered generalised functions (distributions).

The generalised random field now can be defined as the random variable, which has as realizations generalised functions. In other words that mean that probability space can be identified with a space of realizations chosen as a tempered distributions $\mathcal{S}(\mathbb{R}^d)'$.

By a state (or probability distribution of realizations of a generalised random field) we mean a probability measure on the space $\mathcal{S}'$ defined on elements of the σ-algebra $\mathcal{B}$. We

denote by $\mathcal{G}$ the class of all these states. For each state $\mathcal{P} \in \mathcal{G}$ the functional on the space $\mathcal{S}$ is given:

$$X(\varphi) = \int_S \exp(i\Phi_\varphi(f))\mathrm{d}\mathcal{P}(df) \tag{3.2.4}$$

where $\Phi_\varphi(f)$ is a linear functional on the space $\mathcal{S}'$. Equation is called the characteristic functional of the state $\mathcal{P}$. Clearly the characteristic functional assumes the value 1 for $\varphi = 0$ and it is continuous and nonnegative definite. Conversely, any functional on $\mathcal{S}$ having these properties is the characteristic functional of some state $\mathcal{P} \in \mathcal{G}$.

This characteristic function is defined by Bochner-Minlos theorem [81, 124, 137, 138]:

Theorem 3.2.1. *Let X be a function defined on a nuclear space $\mathcal{S}$ such that:*

- *there exist an integer p such that X is continuous relative to the norm $\|\|\|_p$*
- *X is a non-negative function*
- *$X(0) = 1$*

Then there exists an unique probability measure P on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ such that for all $\varphi \in \mathcal{S}$:

$$X(\varphi) = \int_{\mathcal{S}} exp(i\langle\omega, \varphi\rangle)dP(\omega) \tag{3.2.5}$$

where $\langle ., . \rangle$ denotes the duality linear form. This function is called characteristic function.

Afterwards, the definition of a random field can be introduced:

Definition 3.2.2. *A generalised random field is a family of random variables indexed on the space $\mathcal{S}(\gamma, \varphi)$ defined on a probability space $(\Omega, \mathcal{B}, P)$ such that application $\varphi \to \gamma$ is linear and continuous.*

For example the functional satisfying the Bochner-Minlos theorem [81, 82] is described in Holden [72]. There exist an unique probability measure μ_1 on $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$ with the following property:

$$\mathbf{E}[e^{i(.,\phi)}] = \int_{S'} e^{i\langle\omega,\phi\rangle} d\mu_1(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \tag{3.2.6}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $\|\phi\|^2 = \|\phi\|^2_{L^2(\mathbb{R}^d)}$, while $\langle\omega, \phi\rangle = \omega(\phi)$ is the action of $\omega \in \mathcal{S}'(\mathbb{R}^d)$ on $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $\mathbf{E} = \mathbf{E}_{\mu 1}$ denotes the expectation with respect to μ_1. $\mathcal{S}(\mathbb{R}^d)$ is a Schwartz space [82] and $\mathcal{S}'(\mathbb{R}^d)$ is the dual space equipped with the weak-star topology and its called a space of tempered distributions.

A triplet $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)), \mu_1)$ is called one-dimensional white noise probability space, while μ_1 is called the white noise measure. This measure is often called the normalised Gaussian measure on $\mathcal{S}(\mathbb{R}^d)$. In thus defines a Gaussian probability measure P_γ on the space of tempered distributions. This measure is called the d-parameter white noise measure and the corresponding random variable γ is called the d-parameter white noise process ([89]).

The random variable defined by mapping $\gamma \to \langle \gamma, \varphi \rangle$ has a normalised Gaussian measure, where functions φ are functions in $\mathcal{S}(\mathbb{R}^d)$ which are orthonormal in $L^2(\mathbb{R}^d)$. Then d-parameter smoothed white noise is given by mapping $w : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)' \to \mathbb{R}$ i.e. $w = \langle \gamma, \varphi \rangle$.

For all $\varphi_1, .., \varphi_k \in L^2$ the random variables $\langle ., \varphi_1 \rangle, ..., \langle ., \varphi_k \rangle$ defined on this way are jointly Gaussian and they are independent if the φ_i are mutually orthogonal.

3.3 Classification of Random Fields

The random field can be classified in five distinct ways. First classification splits random fields on continuous and discrete, depending on whether random fields have continuous or discrete spatial parameter x (discrete or continuous realization). The second classification is based on whether the space argument is a scalar or vector. Also, we can separate them by the form of the corresponding probability law on Gaussian [15] and non-Gaussian [63] random fields, as well as on homogeneous and non-homogeneous fields [26] or random fields based on memory or with zero-memory. One more way of classification is using covariance. If covariance is given by

$$\operatorname{cov}(x, y) = \begin{cases} \sigma^2, & x = y \\ o, & \text{otherwise} \end{cases} \tag{3.3.1}$$

then random field is uncorrelated. In other cases random field is correlated.
Because of importance in next two chapters we will speak about Gaussian and non-Gaussian random fields.

3.3.1 Gaussian Random Fields

The random field $\gamma(x, \omega)$ on a region $R \subset \mathbb{R}^d$ is called Gaussian random field if all finite distributions are jointly Gaussian [41]. Gaussian random fields have important role in many practical problems. They occur naturally because they are defined by their second-order statistics

- mean value

$$\bar{\gamma}(x) = \mathbf{E}(\gamma(x, \omega)) \tag{3.3.2}$$

- and covariance function

$$\operatorname{cov}_\gamma(x, y) = \mathbf{E}(\gamma(x, \omega) - \bar{\gamma}(x))(\gamma(y, \omega) - \bar{\gamma}(y)), \tag{3.3.3}$$

which must be symmetric and positive semi-definite [1], as well as by properties of Gaussian random variables [129]:

- any subset of jointly Gaussian random variables is also jointly Gaussian,
- any subset of jointly Gaussian random variables conditioned on any other subset of the original random variables is also jointly Gaussian,
- jointly Gaussian random variables that are uncorrelated are also independent,
- linear combinations of jointly Gaussian random variables are also jointly Gaussian, uncorrelated.

Gaussian random fields are very often used in practical cases. Due to the central limit theorem, independent and identically-distributed random variables will be approximately normally distributed (i.e. following a Gaussian distribution) if the random variables have a finite variance i.e. distribution $F(x)$ of x which approaches a normal distribution F_γ with the same mean and variance:

$$F(x) \simeq F_\gamma(\frac{x-\mu}{\sigma}) \tag{3.3.4}$$

as n increases [133]. Or

$$F_z(z) \rightarrow F_\gamma(z) \tag{3.3.5}$$

as $n \rightarrow \infty$.

3.3.2 Non-Gaussian Random Fields

General stochastic processes are completely characterised only when all their joint probability density functions, of all orders, have been specified. The enormous amount of data required to synthesise such a characterisation has been a significant obstacle to the utilisation of non-Gaussian models in the description of physical processes. It is recalled here that Gaussian stochastic processes are completely characterised by their second-order statistics. Given the obvious non-Gaussian character of a number of important physical phenomena, however, the need has always existed for models that allow for a clear deviation from the Gaussian assumption. Such a non-Gaussian behaviour can be introduced into probabilistic models by specifying a hierarchy of lower order joint density functions. Once these have been specified, algorithms must then be developed for synthesising realizations of processes that possess the postulated statistical properties. Occasionally, analytical representations of these processes in forms that are suitable for further mathematical manipulations can also be obtained [149].

[1] Property of symmetric and positive semi-definite covariance function is not so easy to assert especially in higher dimensions.

Many of the available techniques for simulating non-Gaussian processes represent them as nonlinear transformations of some underlying Gaussian processes. Methods for simulating Gaussian processes are therefore at the core of many non-Gaussian simulation algorithms [62, 63, 130]. The identification of the particular nonlinear transformation to apply to the Gaussian process in order to achieve the specified non-Gaussian content has typically involved an iterative procedure that uses either the spectral density function of the process or its correlation function. In the former case, reliance on the frequency domain of properties of the process limits the applicability of these methods to stationary processes, while the latter case has been developed for the class of translation processes which can be represented as monotonic transformations of Gaussian processes. It should also be noted that theoretical results regarding the convergence of the iterative algorithms associated with this method have been developed only for the case of stationary processes. Besides nonlinear transformations of Gaussian processes, methods have been developed for representing non-Gaussian processes as filtered Poisson processes [135]. Thus, both an appropriate Poisson process along with its modulating function must be identified in order to match the target stationary non-Gaussian process. Clearly, all the earlier methods are restricted to the simulation of stationary processes, and the first two methods have not yet been applied to multi-dimensional or multi-variate processes. While conceptual difficulties could prevent the extension to non-stationary processes, it is believed that algorithmic modifications, though not trivial, may enable the extension of these methods to the multi-dimensional and multi-variate cases. These cases are of great relevance to applications as they may enable the modelling of wind pressure or ground motion records that are correlated over space.

The simulation of non-Gaussian processes based on memory less nonlinear transform of an underlying Gaussian process is still an area of active research. Such processes are known as transformation processes.

Standard Gaussian random variable $\mathcal{N}(0,1)$ can be mapped to a random variable with distribution function F_κ by the transformation $F_\kappa^{-1}(\text{erf}(\mathcal{N}(0,1))$, where erf is the Gaussian distribution function [130]. So non-Gaussian random filed can be expressed as:

$$\kappa(x,\omega) = \phi(x,\gamma(x,\omega)) = F_\kappa^{-1} \circ \text{erf}(\gamma(x,\omega)) \tag{3.3.6}$$

where γ is a Gaussian random field with zero mean and with unit variance and where $F_{\kappa(x)}, x \in R$. The second order statistics for the non-Gaussian random field now can be written:

$$\mathbf{E}(\kappa(x,\omega)) = \int_R \phi(x,\omega) dF_\gamma(w) \tag{3.3.7}$$

where the dF_γ is the standard Gaussian measure with zero mean and unit variance.
The covariance of random field can be obtained :

$$\text{cov}_\kappa(x,y) = \int_R \int_R \phi(x,w_1)\phi(x,w_2) dF_{\gamma(x),\gamma(y)}(w_1,w_2) - \mu_\kappa(x)\mu_\kappa(y) \tag{3.3.8}$$

where $F_{\gamma(x),\gamma(y)}$ is the joint probability density of the two random variables $\gamma(x)$ and $\gamma(y)$. Because one has the second order statistics of non-Gaussian random field, not of Gaussian,

ϕ and $\text{cov}_\gamma(x, y)$ must be selected so that κ satisfies given second order statistics of $\mu_\kappa(x)$ and $\text{cov}_\kappa(x, y)$.
The other way is to represent the non-Gaussian random field as the finite sum of independent centred random variables times deterministic functions:

$$\kappa(x, \omega) = \bar{\kappa} + \sum_{i=1}^{M} k_i(x)\kappa_i(\omega). \tag{3.3.9}$$

The advantage of this representation is that the field is described in a finite number of random variables. For more in-depth discussions about Gaussian and non-Gaussian random fields see e.g. text book [63] and paper [64].

3.3.3 α-stable Random Variables

α-stable random variables may be used to model numerically random fields with infinite variance, but they will be discussed very shortly here. For introduction see [21, 63, 150].

The α-stable distribution [63], which can model phenomenon of an impulsive nature, is a generalisation of the Gaussian distribution and is appealing because of two main reasons [52]: first, it satisfies the stability property distribution and second satisfies the generalised central limit theorem. There is no closed-form expression for the probability density function of α-stable distributions, but only it's given the characteristic function [52].

So α-stable random variables are characterised by the characteristic exponent, which lies in the domain $(0, 2]$. The characteristic exponent controls the heaviness of the tails of the density function. The tails are heavier, and thus the noise more impulsive, for low values of α while for a larger the distribution has a less impulsive behaviour. The case of $\alpha = 2$ corresponds to the Gaussian distribution, while $\alpha = 1$ corresponds to the Cauchy distribution i.e. non-Gaussian distribution.

The class of α-stable distributions does not possess finite second (or higher) moments. In fact, α -stable distributions with $\alpha \neq 2$ have finite moments only for order lower than α.

The class of α-stable random variables is characterised by the property that linear combinations of α-stable random variables are again α -stable random variables. Also the family of α-stable random distributions contains all limiting distributions of sums of random variables. This simplifies the handling of series expansions with α-stable random variables as coefficients as the resulting marginal distributions are known.

3.4 Properties of Random Fields

The study of the behaviour of the random field such as continuity, differentiability and integrability can lead to conclusion regarding the geometry of the random field and con-

sequently the regularity, homogeneity. Detailed information about all properties are given in [26].

3.4.1 Stochastic Convergence

When someone speaks about convergence then there are four different ways to speak about it. Let κ_n be a sequence of random variables of $L^2(\Omega, \mathcal{B}, P)$, then it is said that κ_n converge to the random variable κ [26]:

- in a mean square sense if
$$\lim_{n\to\infty} \mathbf{E}\,|\kappa_n - \kappa|^2 = 0 \tag{3.4.1}$$
- almost surely or with probability one if
$$P[\lim_{n\to\infty} \kappa_n = \kappa] = 1 \tag{3.4.2}$$
- in probability if for all $\varepsilon > 0$
$$\lim_{n\to\infty} P[|\kappa_n - \kappa| > \varepsilon\ ] = 0 \tag{3.4.3}$$
- weakly or in distribution if
$$\lim_{n\to\infty} F_{\kappa_n}(k) = F_\kappa(k) \tag{3.4.4}$$

on the continuity set of F_κ. But all these types are related between, so both the convergence in the mean square sense and almost surely convergence , lead us to convergence in probability which leads to the convergence in distribution.

3.4.2 Stochastic Continuity

A random field is continuous [26] in the mean square sense if

$$\kappa(x+h) \to \kappa(x) \tag{3.4.5}$$

as $h \to 0$. It can be shown that if

$$\mathbf{E}(|\kappa(x+h) - \kappa(x)|^\lambda \leq \frac{\alpha\,|h|^{2n}}{|log\,|h||^{1+\beta}} \tag{3.4.6}$$

where α is a positive constant, $\lambda > 0$ and $\beta > \lambda$, then the random field is almost surely continuous over the compact set $C \subset \mathbb{R}^n$.

3.4.3 Stochastic Differentiation

A random field is differentiable [26] in the mean square sense with the respect to x_i and that property is called partial derivative of $\kappa(x)$ at x and it's given by:

$$\frac{\partial\kappa(x)}{\partial x_i} = \lim_{h\to 0}\frac{\kappa(x+h\varepsilon_i)-\kappa(x)}{h}, \tag{3.4.7}$$

meaning that

$$\lim_{h\to 0}\mathbf{E}[\frac{\kappa(x+h\varepsilon_i)-\kappa(x)}{h}-\frac{\partial\kappa(x)}{\partial x_i}]^2 = 0. \tag{3.4.8}$$

The random field is differentiable if and only if the mean value is differentiable and the covariance

$$\mathrm{cov}(\frac{\partial\kappa(x)}{\partial x_i},\frac{\partial\kappa(x')}{\partial x_i'}) = \frac{\partial^2 c_x(x,x')}{\partial x_i\partial x_i'} \tag{3.4.9}$$

exists and it's finite at all diagonal points $x = x'$. Since we are assuming zero mean, random field condition for the mean value may be relaxed.

If condition for the continuity is satisfied in the case of the mean square partial derivatives of a random field then random field is almost sure differentiable random field.

3.4.4 Homogeneity

The random process is called homogeneous in the wide sense if

$$\mathbf{E}(x) = \mathrm{const}, \tag{3.4.10}$$

$$\mathrm{cov}(x,y) = c(x-y) \tag{3.4.11}$$

that is its mean value is a constant, and its covariance depends only on the vector distance between two points in space [26].

The homogeneity property amounts to the fact that there exist in the closed linear subspace H spanned by the random variable x in $L^2(\Omega,\mathcal{B},P)$ a group of unitary operators U_h such that

$$U_h\kappa(x) = \kappa(x+h) \tag{3.4.12}$$

where x, h belongs to $\mathbb{R}^n$. If one talks about the physical meaning of homogeneity that is the macrostructure of the underlying physical variate do not change over the space.

Also, we can speak about the homogeneity in the strict sense when all probability distributions remain the same under the translations. But homogeneity in the wide sense doesn't imply homogeneity in the strict sense. This can be said only for Gaussian random fields. In addition the probability in the strict sense is very rarely used in practice so under the word homogeneity we assume homogeneity in the wide sense.

In one-dimensional case this condition is equivalent to the assumption of the stationary random field.

Homogeneous random fields can be represented by their spectral representation. Applying the Fourier transform one obtains the covariance function:

$$c_{x(h)} = \int_{R^n} \exp[iwh] dQ_x(w). \tag{3.4.13}$$

If one employ the Bochner theorem [139] then it will obtain the function $c_x(h)$ as a non-negative definite only if expression (3.4.13) states. Hence the Fourier transform is the transform of a bounded non-negative function, then if symmetry is fulfilled, function $\mathrm{cov}(x, y) = c(x - y)$ is a covariance function [89].

3.4.5 Isotropy

A random field is called isotropic in strict sense if its all distribution functions are invariant under the orthogonal transformations. The definition of isotropy in wide sense doesn't imply isotropy in the strict sense. This can be said only for Gaussian random fields [26].

If $\kappa(x)$ is isotropic random field then it is an homogeneous random field, although the converse is not generally true. Permissible covariance i.e. nonnegative-definite function is the continuous function $c_x(r)$, where $r = |h| \in \mathbb{R}^n$ such that holds:

- $$c_x^{'}|_{r=0} < 0 \tag{3.4.14}$$
- $$\lim_{r\to\infty} \frac{c_x(r)}{r^{\frac{(1-n)}{2}}} = 0 \tag{3.4.15}$$
- $$c_x^{"}(r) \geq 0 \quad \text{in} \quad \mathbb{R} \tag{3.4.16}$$
- $$\int_r^\infty \frac{u}{\sqrt{u^2 - r^2}} dc_x^{"}(r) \geq 0 \quad \text{in} \quad \mathbb{R}^2 \tag{3.4.17}$$
- $$c_x^{"}(r) - r c_x^{'''}(r) \geq 0 \quad \text{in} \quad \mathbb{R}^3. \tag{3.4.18}$$

Geometrical anisotropy is characterised by:

$$c_x(h) = c_x(\sqrt{h^T \mathbf{G} h}) \tag{3.4.19}$$

where $\mathbf{G}$ is a non-negative matrix. In the case when $\mathbf{G}$ is the identity matrix then one obtains isotropic random field.

3.4.6 Ergodicity

A random field is ergodic if the mean and the covariance of $\kappa(x)$ coincide with those calculated by means of the single available realization. More about this is given in [26, 124].

3.5 Homogeneous Random Fields

A random field is called homogeneous if its finite dimensional distributions are invariant under the translations as it's said. In other words a random field is considered homogeneous when its mean value and variance do not depend on the location [132] and its correlation is dependent solely on the distance between the variables i.e. its covariance function is a function of separation distance only :

$$\mathrm{cov}_\kappa(x, y) = c(x - y). \tag{3.5.1}$$

Every Gaussian random field is a homogeneous but the converse does not hold.

A random field $\kappa(x)$ defined on a homogeneous space $R = \{x\}$ of points x equipped with a transitive transformation group $G = \{g\}$ of mappings of R into itself, and having the property that the values of the statistical characteristics of this field do not change when elements of G are applied to their arguments is called homogeneous random field [187, 188, 189]. One distinguishes two different classes of homogeneous random fields: $\kappa(x)$ is called a homogeneous random field in the strict sense if for all n=1,2,... and $g \in G$, the finite-dimensional probability distribution of its values at any n points $\{x_1, x_2, .., x_n\}$ coincides with that of its values at $\{gx_1, gx_2, ..gx_n\}$. If $\mathbf{E}\left|\kappa(x)\right|^2 < \infty$ and $\mathbf{E}(\kappa(x)) = \mathbf{E}(\kappa(gx)), \mathbf{E}(\kappa(x)\kappa(x_1)) = \mathbf{E}(\kappa(xg)\kappa(x_1g))$ for all $(x, x_1) \in R$ and $g \in G$, then $\kappa(x)$ is called a homogeneous random field in the wide sense.

An important special case is homogeneous random field on a k dimensional Euclidean space $\mathbb{R}^k$(or on the lattice $\mathbb{Z}^k$ of points of $\mathbb{R}^k$ with integral coordinates), where G is the group of all parallel translations. Sometimes the term "homogeneous random field" is reserved for a field of this type. A homogeneous random field on $\mathbb{R}^k$, with G the group of all isometric transformations of $\mathbb{R}^k$(generated by parallel translations, rotations and reflections) is often called an isotropic homogeneous random field.

The concept of a homogeneous random field is a natural generalisation of a stationary stochastic process: in both cases, the field and the covariance function admit a spectral decomposition of special kind. Homogeneous random fields and some of their generalisations often arise in questions of an applied nature. In particular, in the statistical theory of turbulence, an important role is played by isotropic homogeneous (scalar and vector) random fields on $\mathbb{R}^k$, as well as by so-called simultaneously locally homogeneous and locally isotropic random fields (that is, fields with homogeneous and isotropic increments), which are simple generalisations of isotropic homogeneous fields. Moreover, in the modern theory of physical quantum fields and in statistical physics there are applications of the theory of

generalised homogeneous random fields [136], which include homogeneous random fields as a special case [187, 188, 189].

3.6 Maximum Entropy Distribution

The applications of the maximum entropy distribution [133] are used in the case when one deals with problems involving the determination of unknown distributions. Available informations are in the form of known expected values or other statistical functionals, so the solution is based on the principle of the maximum entropy. The principle of maximum entropy is based on the fact that we determine the unknown distributions maximising the entropy subject to a given constraints. Second problem arises when it is given source entropy and it is solved by constructing various random variables with goal to minimize their expected values.

The entropy of the random variable is defined by [30, 78, 101] :

$$H(\kappa) = \int_{-\infty}^{+\infty} p(x)\log(\frac{1}{p(x)})dx \tag{3.6.1}$$

where $p(x)\log(\frac{1}{p(x)})$ is understood to be zero if $p(x) = 0$.
Knowing the principle of maximum entropy we can make next conclusions:

- therefore, if all you know about a distribution is its mean and standard deviation, it is often reasonable to assume that the distribution is normal;
- the uniform distribution on the interval $[a, b]$ is the maximum entropy distribution among all continuous distributions which are supported in the interval $[a, b]$ (which means that the probability density is 0 outside of the interval);
- if the available informations are the second-order statistics then a Gaussian distribution should be chosen.

3.7 Models of Correlation Functions

- Isotropic correlation function
  As we said previous, isotropic correlation function is given as:

$$\mathrm{cov}(x, y) = c(\|x - y\|). \tag{3.7.1}$$

  Detailed description of models of correlation functions of homogeneous and isotropic random fields is given in text book by Semenovska [159].

- Ellipsoidal correlation structure

  By scaling the axes of an isotropic process one obtains a process with an ellipsoidal covariance function [175].

- Gaussian type
  Very frequently used model is a Gaussian type of covariance :

$$\mathrm{cov}(r) = \sigma^2 \exp(-a^{-2} r^2), \tag{3.7.2}$$

  which associated process is smooth i.e. continuous and differentiable of any order [89].

- Autoregressive correlation models [26, 175, 183]

  In the case of the first order difference equation

$$\kappa_{t+1} = a\kappa_t + u_t, \tag{3.7.3}$$

  where κ is a random variable and u_t is an uncorrelated random series, the associated covariance function is:

$$\mathrm{cov}(r) = \sigma_x^2 \exp^{-\alpha|r|} \tag{3.7.4}$$

  , where r is a distance and :

$$\sigma^2 = \frac{G_0 \pi}{2\alpha}, \tag{3.7.5}$$

  where G_0 is the constant spectral density of u_t. Previous equation is in fact the Langevin equation arising in the analysis of Brownian motion. For spatial process on the line one has the bilateral autoregression in distinct of previous equations which are unilateral autoregression [178] :

$$\kappa_t = a_1\kappa_{t-1} + a_2\kappa_{t+1} + u_t, \tag{3.7.6}$$

  where the associated covariance function is given by:

$$\mathrm{cov} = \sigma_x^2(\alpha|r| + 1) \exp^{-\alpha|r|}. \tag{3.7.7}$$

  Whittle [178] shown that it may seem unnecessary to introduce the bilateral type of scheme when any such scheme may effectively be reduced to an unilateral one. When we consider how much more complicated and indeterminate the parameter estimates are for a bilateral model the step of reducing is needed but for the two-dimensional case the reduction to an unilateral scheme complicates matters very much, in contrast to the one-dimensional case we have just considered.

  In two dimensions, the exponential correlation function can be written as $C(r) = \exp^{-r/b}$ where r is the distance between two spatial points. This function has been also used in the literature. However, as Whittle pointed out in [178], it is necessary to

introduce autoregression schemes with dependence in all directions for more realistic models of random series in space. The simplest such model is

$$\kappa_{xt} = a(\kappa_x, \kappa_{x+1,t} + \kappa_{x-1,t} + \kappa_{x,t+1} + \kappa_{x,t-1}) + u_{xt}, \tag{3.7.8}$$

where κ_{xt} is a random field at grid (x,t) and u_{xt} is independent identically distributed random field. This model corresponds to stochastic Liouvile equation in the continuous case

$$[(\frac{\partial}{\partial x}^2 + \frac{\partial}{\partial y}^2 - \frac{1}{(\frac{1}{a} - 4)})]\kappa(x,y) = u(x,y). \tag{3.7.9}$$

The 'elementary' correlation function in two dimensions can be solved from the above equation:

$$\mathrm{cov}(r) = \frac{r}{b} K_1(\frac{r}{b}) \tag{3.7.10}$$

where K_1 is the modified Bessel function of the second kind with order 1, b scales as the correlation length and r is the distance between two points.

3.8 Random Field Modelling of Natural Processes

By modelling natural process using random models one develops model, studies its behaviour and uses to make predictions about the natural process. For using the application of spatial random fields it is necessary to make certain hypotheses. Hypotheses lead in methodology are hypotheses where locally the natural process is viewed as the random variable but globally it is characterised by spatial dependence of the adjacent random variables that constitute the spatial random field. Among various parameters of a spatial random field model it's chosen the statistical moments of order up to two which make the SRF (spatial random field) representation of a natural process feasible. One realization of the natural process contains all statistical information about ensemble. These auxiliary hypotheses include homogeneity, isotropy and intrinsity of some order. For developed model, process of examining gives us conclusions about behaviour of the natural process. For that it is used semivariogram [89]:

$$v(h) = \mathrm{var}(\kappa(x) - \kappa(x+h)) \tag{3.8.1}$$

where $v(x)$ is the semivariogram [166], and $h \in \mathbb{R}^d$ is the lag vector. It is simple to see that it depends of the difference between samples at different places depending of h. The semivariogram is similar to the covariance with difference that doesn't depend of the measured mean.

Anisotropic properties of a field are shown on the base of the variation of the semivariogram along the several directions in space. If it varies significaly along different directions the SRF is called anisotropic i.e. natural process is anisotropic. Observing the range of

the semivariogram one can get conclusion about the zone of influence of local value of the natural process. The behaviour on the large distances destine the degree of homogeneity. Also, on the base of the behaviour of the origin of semivariogram one can get information about the regularity in the spatial variation in the process. Regularity depends of used sampling scale. Often there is an appearance of the $v(0) \neq 0$ known as the nugget effect [88]. The reasons for its appearance are incorrect sampling or discontinuities of the material. When a nugget effect is there the random field is irregular and it might be to model with the generalised random field. Mathematically speaking the nugget effect is a limit of the variation of differences between lagged samples taken closer and closer together [166]. Estimation of the nugget effect can provides valuable information about the process.

3.9 Conclusion

To modell random fields one need their statistical informations obtained by measurements. To obtain these informations in the case of spatial random field is very difficult because standard methods or time series analysis cannot be directly applied to spatial data (i.e. there may be different trends in different directions). A valid probabilistic model of a physical quantity cannot be obtained from experiments alone. Therefore, some hypotheses are defined. These hypotheses are approximately justified like stationarity, ergodicity, or validity of Gaussian distributions for material properties. The study of behaviour of the random field such as continuity, differentiability and integrability can lead to conclusion regarding the geometry of the random field and consequently the regularity, homogeneity.

When available informations of the random field are in the form of known expected values or other statistical functionals, determination of the unknown distribution is based on the maximum entropy principle. If it is known that variable takes value in bounded region, it should be modelled by uniform probability distribution, while in the case of second order statistics Gaussian distribution should be chosen. For higher moments also one can use mentioned principle.

On the other hand most publications of stochastic finite element techniques do not employ real life statistical models. Very often it is used exponential model of a Gaussian type given by Eq.(3.7.2). Hence, as it is pointed out this type of model will not fit for the case of two or three spatial dimensions. In the end, all informations for the good model depend on the application .

For some informations about the process of validation and verification in reliability see paper of Babuska and coworkers [8].

Chapter 4

Representation of the Random Fields

In last years the spectral stochastic finite element approach obtained so much attention from the scientists. This method deals with random fields using polynomial chaos variables.

Whole idea comes from the fact of weak sense stationary random process [120] characterised by its spectrum i.e. Fourirer transform. The random process $s(t,\omega)$ is a weak sense stationary or weakly stationary if its mean is independent of time t and the covariance is a function of the time lag i.e. $\mathrm{cov}(t_1,t_2) = c_s(t_1 - t_2)$. The Fourirer transformation of the time-lag covariance function is then given:

$$S(\nu_k) = \int_0^T c_s(t)\exp(-i2\pi\nu_k t)dt, \quad \nu_k = \frac{k}{T}; k \in \mathbb{Z} \tag{4.0.1}$$

with the obvious modifications if the time interval is infinite. In this case processes may be synthesised from its spectrum:

$$s(t,\omega) = \overline{s}(t) + \sum_{k=-\infty}^{\infty} \xi_k(\omega)\sqrt{S_s(\nu_k)}\exp(i2\pi\nu_k t) \tag{4.0.2}$$

where the $\xi_k(\omega)$ are uncorrelated random variables of unit variance and vanishing mean.

The description of random fields where we assign a random variable to each point in some region R in space is in many ways similar to the case of stochastic processes. The only difference is that the "index set" is changed from the time interval to space R.

4.1 The Karhunen-Loève Expansion

The Karhunen-Loève expansion (KLE) also known as the proper orthogonal decomposition or the method of empirical eigenfunctions, has been widely discussed in many books and

papers [18, 19, 27, 57, 58, 59, 74, 86, 144, 147, 168]. The principal idea behind KLE expansion method is that, given an ensemble of data, one can find a basis of a given dimension that spans that data optimally in L^2 sense. It was invented independently by Karhunen (1947), Loève (1948) and Kac and Siegert (1947).

Let $\kappa(x,\omega)$ be a random field (function of position vector x defined over the domain R, with ω belonging to the space of random events Ω) and let $\mathrm{cov}_\kappa(x,y)$ be its admissible covariance function, defined as:

Definition 4.1.1. *A covariance function $cov_\kappa(x,y) \in L^2$ is said to be admissible if it is symmetric and positive definite in the sense that*

$$0 \leq \sum_{k=1}^{n}\sum_{j=1}^{n} a_k cov_\kappa(x_k,x_j)a_j, \forall x_k, x_j \in \mathbb{R}, a_k, a_j \in \mathbb{C}. \tag{4.1.1}$$

The basic results and examples on positive definite functions are given in [16, 153].

Given an admissible covariance function in the sense of the previous definition, the associated covariance operator T is a symmetric, non-negative and compact integral operator defined as:

$$T : L^2(R) \to L^2(R) \tag{4.1.2}$$

with the covariance as kernel:

$$(Tu)(x) = \int_R \mathrm{cov}_\kappa(x,y)u(y)dy. \tag{4.1.3}$$

Admissible covariance function by definition has all eigenfunctions mutually orthogonal forming a complete set, which spans function space to which $\kappa(x,\omega)$ belongs. They are given by eigenvalue problem:

$$T\kappa_i = \lambda_i\kappa_i, \kappa_i \in L^2(R), i \in \mathbb{N}, \tag{4.1.4}$$

which represents a homogeneous Fredholm integral equation of a second kind, where operator T is a Fredholm operator.

As it is mentioned, the covariance is bounded, symmetric and positive definite. This simplifies the analysis because eigenvalues and eigenfunctions, as a solution of a given equation, have some very usefully properties [59]:

- the set of continuous eigenfunctions κ_i is orthogonal and complete,
- for each eigenvalue λ_i there correspond at most finite number of linear independent eigenfunctions,
- the eigenvalues are real and positive,
- the eigenvalues have a property :

$$\|\mathrm{cov}_\kappa\|^2_{L^2(RxR)} = \int_R\int_R |\mathrm{cov}_\kappa(x,y)|^2\, dxdy = \sum_i \lambda_i^2, \tag{4.1.5}$$

- the kernel admits of the uniformly convergent expansion:

$$\mathrm{cov}_\kappa(x,y) = \sum_{i=1}^{\infty} \lambda_i \kappa_i(x)\kappa_i(y). \tag{4.1.6}$$

Therefore, one has a countable number of eigenpairs (κ_i, λ_i) where the sequence of KL-eigenvalues λ_i is enumerated with decreasing magnitude and is either finite or tends to zero as $i \to 0$ i.e. $\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq 0$ (with multiplicity continued). KL-eigenfunctions are assumed to be L^2- orthonormal i.e.

$$\int_R \kappa_m(x)\kappa_n(x)dx = \delta_{mn}, \quad m,n = 1,2,.. \tag{4.1.7}$$

The Karhunen-Loève expansion of random field $\kappa(x,\omega)$ with finite mean and covariance is given by:

$$\kappa(x,\omega) = \bar{\kappa}(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\omega)\kappa_i(x) \tag{4.1.8}$$

where $\bar{\kappa}(x)$ denotes the mean value of the process, $\xi_i(\omega)$ are set of standard uncorrelated normal random variables and λ_i and κ_i are the eigenvalues and eigenfunctions of the covariance kernel respectively.

The standard uncorrelated normal random varaibles $\xi_i(\omega)$ can be found by multiplying equation (4.1.8) by $\kappa_i(x)$ and integrating over the domain R [59]:

$$\xi_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_R \kappa(x,\omega)\kappa_i(x). \tag{4.1.9}$$

In practice instead of the equation (4.1.8) it is used truncated form given by m terms:

$$\widehat{\kappa}(x,\omega) = \bar{\kappa}(x) + \sum_{i=1}^{m} \sqrt{\lambda_i}\xi_i(\omega)\kappa_i(x), \tag{4.1.10}$$

defining truncating error.
The Karhunen-Loève expansion possesses other interesting properties [168]:

- due to non accumulation of eigenvalues around a non zero value, it is possible to order them in a descending series converging to zero. Truncating the ordered series after the M-th term gives the KL approximated field.

- The covariance eigenfunction basis is optimal in the sense that the mean square error (integrated over) resulting from a truncation after the M-th term is minimized (with respect to the value it would take when any other complete basis is chosen).

- From equation (4.1.6), the error variance obtained when truncating the expansion after M terms turns out to be, after basic algebra :

$$\mathrm{var}(\kappa(x,\omega)-\hat{\kappa}(x,\omega))=\sigma^2(x)-\sum_{i=1}^{M}\lambda_i\kappa_i^2=\mathrm{var}[\kappa(x,\omega)]-\mathrm{var}[\hat{\kappa}(x,\omega)]. \tag{4.1.11}$$

The right-hand side of the above equation is always positive because it is the variance of some quantity. This means that the Karhunen-Loève expansion always under-represents the true variance of the field. The most important aspect of this spectral representation is that spatial random fluctuations have been decomposed into a set of deterministic functions in spatial variables multiplying random coefficients that are independent of these variables. If the random process being expanded is Gaussian, then the random variables form an orthonormal Gaussian vector. The Karhunen-Loève expansion is mean square convergent irrespective of the probabilistic structure of the process being expanded. It can be shown that the eigenvalues of the covariance kernel decay monotonically with increasing value of their index. The monotony of this decay is guaranteed by the symmetry of the covariance function, and its rate is related to the rate of decay of the Fourier transform of the correlation function of the process being expanded.

4.2 Resolution of the Integral Eigenvalue Problem

Fredholm equation can be solved analytically only for few autocovariance functions and geometries of. Detailed closed form solutions for triangular and exponential covariance functions for one-dimensional homogeneous fields can be found in [59], where $R=[-a;a]$ (a is a real number). Extension to two-dimensional fields defined for similar correlation functions on a rectangular domain can be obtained as well. Our ultimate goal is to obtain a representation in mutually independent random variables. As we said, for Gaussian random fields $\kappa(x,\omega)$ the KL-expansion directly yields such a representation: we have that uncorrelated random variables $\xi_i(\theta)$ are Gaussian and mutually independent. But in the case of non-Gaussian random variables, random variables ξ_i are uncorrelated but not mutually independent [88]. Also, in that case probability distributions of the ξ_i are then not analytically known. However, they may be obtained numerically eq.(4.1.9) [74].

The Rosenblatt transform is a standard technique for computing representation of non-Gaussian random fields in independent random variables. This general technique requires a representation of the joint probability distributions of the $\{\xi_1,\xi_2,...\}$ as products of conditional probability distributions.

Let $\gamma(x,\omega)$ be a Gaussian random field which KL-expansion is given by equation (4.1.8) with mutually independent standard Gaussian RVs $\theta=(\theta_1,\theta_2,...)$. The dependence on ω is omitted and random field is written as:

$$\gamma(x,\omega)=\bar{\gamma}(x)+\sum_{i=1}^{m}\sqrt{\lambda_i}\theta_i\gamma_i(x). \tag{4.2.1}$$

The non-Gaussian random field can be represented as transformation of Gaussian random

field [133]:

$$\kappa(x,\omega) = \phi(x,\gamma(x,\omega)), \tag{4.2.2}$$

so that one can write:

$$\kappa(x,\omega) = \phi(x,\bar{\gamma}(x) + \sum_{i=1}^{m} \sqrt{\lambda_i}\theta_i\gamma_i(x)), \tag{4.2.3}$$

where again the dependence of ω is omitted.

4.3 Random Field Discretization

The discretisation of a spatial distributed random field is done by specifying the set of a correlated random variables such that each random variable represents the random field over a particular element of a structure [66]. This type of discretisation represent the stochastic spatial variation of any parameter of a structure. There are several methods for the process of discretisation of random fields into random variables. But one can distinct three basic approach:

- point discretisation, where the random variables are selected values of random field at some given points i.e. values of random field at midpoint of element (the midpoint method , Der Kiureghian and Kee [92] or at nodes of element (called shape function method Liu at al. [102, 103] or at each Gauss point of the element (the integral point method, Matthies et al. [116] referring to Brenner and Bucher [17] etc.

- average discretisation, where the random variables are weighted integrals over domain of element Ω_e

$$\xi_i = \int_{\Omega_e} \kappa(x)\omega(x)d\Omega. \tag{4.3.1}$$

  Some of these methods are spatial average proposed by Vanmarcke and Grigoriu [175, 176] defining the approximated field in each element as a constant being computed as the average of the original field over the element, then the weighted integral method developed by Deodatis [35, 36], Deodatis and Shinozuka [37] and also investigated by Takada [169, 170] in the context of stochastic finite elements.

- series expansion methods, where the field is exactly represented as series of involving random variables and deterministic spatial functions. The approximation is then obtained as a truncation of the series. Some of used series expansions are the Karhunen-Loève and Newmann expansion.

4.4 The Discrete Karhunen-Loève Expansion

The eigenvalue problem needs to be solved numerically and for that one can use a standard techniques. The Reyleigh-Ritz method is used to discretise the Karhunen-Loève eigenvalue problem by projecting onto finite-dimensional subspace [88].

Let in space V^h is given a basis $N_1(x), N_2(x), ..., N_n(x)$, then eigenvalues are computed by:

$$\kappa_i^h(x) = \sum_{j=1}^{n} N_j(x)\kappa_{ji}^h = \mathbf{N}(x)\kappa_i^h. \tag{4.4.1}$$

Let $\mathcal{P}_{V^h}$ be the projection onto n-dimensional space V^h, then the eigenvalues of equation (4.1.3) can be found in $\mathcal{P}_{V^h}$ as:

$$\mathbf{W}\kappa_i^h = \lambda_i^h \mathbf{M}\kappa_i^\kappa \tag{4.4.2}$$

where $\mathbf{W}$ is a symmetric positive semi-definite matrix:

$$W_{ij} = \int_R \int_R N_i(x)\mathrm{cov}_\kappa(x,y)N_j(y)dxdy \tag{4.4.3}$$

and $\mathbf{M}$ is a Gram matrix also symmetric and semi-positive:

$$M_{ij} = \int_R N_i(x)N_j(y)dxdy. \tag{4.4.4}$$

For solving this equation one can use standard techniques. This is standard eigenvalue problem and can be solved by techniques given in [134, 146, 180].

4.5 Polynomial Chaos Decomposition

The covariance function of the solution process is not known a priori, and hence the KL expansion may not be used to approximate it. Furthermore, even when the problem specification only involves Gaussian parameters or processes, the solution process is not necessarily Gaussian, so that the KL representation may not be a suitable approximation even when much is known about the covariance function of the solution. Thus, an alternative representation means is needed, and the polynomial chaos decomposition addresses this need [1, 33, 53, 55, 59, 76, 84, 95, 109, 154, 168, 185, 147].

Under specific conditions [154], a stochastic process can be expressed as a spectral expansion in terms of suitable orthogonal eigenfunctions with weights associated with a particular density. A well-studied example is the Wiener process (Brownian motion) which can be written as a spectral expansion in terms of Hermite polynomials and the Normal distribution. Other examples include Charlier polynomials and the Poisson distribution, and the Laguerre polynomials and the Gamma distribution [154].

The polynomial chaos is also called the Wiener polynomial chaos, the Wiener chaos or the Wiener Ito Chaos. The word chaos used by N.Wiener, has a nothing to do with modern usage of the word in mathematics, where it characterises the unpredictable behaviour of dynamical systems.

Let us first introduce Hermite polynomials and its algebra, so we can speak about polynomial chaos expansion.

4.5.1 Hermite Polynomials

Let consider a real space $\mathbb{R}$, equipped with the Gaussian measure $dP(\xi) = p(\xi)d\xi$, where $p(\xi) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}$ and where $d\xi$ is a Lebesgue measure [109]. Now one can introduce the space of square integrable functions:

$$L^2(\mathbb{R}, P) = \{f(\xi); \int_{-\infty}^{\infty} f(\xi)^2 dP(\xi) \leq \infty\}. \tag{4.5.1}$$

The inner product now can be defined as

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(\xi)g(\xi)dP(\xi) = \int_{-\infty}^{\infty} f(\xi)g(\xi)P(\xi)d\xi. \tag{4.5.2}$$

Let recollect the equation for the expectation of the random variable:

$$\mathbf{E}[f(\xi)] = \int_{-\infty}^{\infty} f(\xi)^2 dP(\xi). \tag{4.5.3}$$

From previous two equations we can conclude that if ξ is a standard Gaussian random variable with the distribution $\mathcal{N}(0, 1)$, then:

$$\langle f, g\rangle = \mathbf{E}[f(\xi)g(\xi)]. \tag{4.5.4}$$

So Hilbert space also can be defined as a space of a functions of a unit Gaussian random variable with a finite variance. Now when the Hilbert space is defined one can define specific kind of orthogonal polynomials called Hermite polynomials. Let observe generating function of Hermite polynomials [109]

$$\phi(\xi, z) = e^{-\frac{z^2}{2}+\xi z}, \tag{4.5.5}$$

and expand this function in Taylor series of variable z (treating ξ as a parameter)

$$\phi(\xi, z) = \sum_{n=0}^{\infty} \frac{\partial^n \phi(\xi, z)}{\partial z^n}|_{z=0}(\frac{z^n}{n!}). \tag{4.5.6}$$

Also

$$\phi(\xi, z) = e^{\frac{\xi^2}{2}}e^{-\frac{(z-\xi)^2}{2}} = e^{\frac{\xi^2}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{d\xi^n}(e^{-\frac{\xi^2}{2}})z^n = \sum_{n=0}^{\infty} \frac{\mathcal{H}_n(\xi)}{n!} z^n, \tag{4.5.7}$$

where with $\mathcal{H}_n(\xi)$ are denoted the Hermite polynoms i.e. un-normalised polynoms. As we can notice they are the coefficients of Taylor expansion

$$\mathcal{H}_n(\xi) = \frac{\partial^n \phi(\xi, z)}{\partial z^n}|_{z=0}. \tag{4.5.8}$$

given in explicit form [1]

$$\mathcal{H}_n(\xi) = (-1)^n e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n}(e^{-\frac{\xi^2}{2}}), \quad n = 0, 1, 2, ... \tag{4.5.9}$$

Differentiating previous equation we get

$$\mathcal{H}_n^{'}(\xi) = \xi \mathcal{H}_n(\xi) - \mathcal{H}_{n+1}(\xi). \tag{4.5.10}$$

Differentiating both sides of the equation after the Taylor expansion and shifting the summation index and writing equation for the z^n, we will have

$$\mathcal{H}_n^{'}(\xi) = n\mathcal{H}_{n-1}. \tag{4.5.11}$$

Combining these two equations (4.5.9) and (4.5.10), we get the recursive relation for the un-normalised Hermite polynoms

$$\mathcal{H}_{n+1}(\xi) - \xi \mathcal{H}_n(\xi) + n\mathcal{H}_{n-1}(\xi) = 0, \quad n = 1, 2, ... \tag{4.5.12}$$

with $\mathcal{H}_{-1}(\xi) = 0, \mathcal{H}_0(\xi) = 1$. Using recursive relations we get first few un-normalised Hermite polynoms:

$$\mathcal{H}_0(\xi) = 1, \mathcal{H}_1(\xi) = \xi, \mathcal{H}_2(\xi) = \xi^2 - 1, .. \tag{4.5.13}$$

Hermite polynomials posses property of orthogonality with respect to Gaussian measure:

$$\langle \mathcal{H}_n, \mathcal{H}_m \rangle = \mathbf{E}[\mathcal{H}_n(\xi)\mathcal{H}_m(\xi)] = n!\delta_{nm}. \tag{4.5.14}$$

Although, in practice very often are used the normalised Hermite polynomials which are defined as

$$h_n(\xi) = \frac{\mathcal{H}_n(\xi)}{\sqrt{(n!)}} = (n!)^{-1/2}(-1)^n e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n}(e^{-\frac{\xi^2}{2}}). \tag{4.5.15}$$

The normalized Hermite polynomials form a complete orthonormal basis in the Hilbert space L^2 and their mathematical expectation is given

$$\mathbf{E}[h_n(\xi)] = \int_{-\infty}^{\infty} h_n(\xi) dP(\xi) = 0 \quad if \quad n \neq 0. \tag{4.5.16}$$

[1] Here it is used generating function for probabilistic Hermite polynomials not the generating function for physics Hermite polynomials, which are given by equation $\mathcal{H}_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n}(e^{\xi^2})$. As we can notice there is a difference in a factor $2^{n/2}$. Here it is used probabilistic definition . It is natural because $e^{\xi^2/2}/\sqrt{2\pi}$ is the probability density for the normal distribution.

Recursive relation for normalised Hermite polynomials becomes

$$\sqrt{n+1}h_{n+1}(\xi) - \xi h_n(\xi) + \sqrt{n}h_{n-1}(\xi) = 0, \tag{4.5.17}$$

with

$$h_{-1}(\xi) = 0, h_0(\xi) = 1. \tag{4.5.18}$$

If ξ is a just single random variable, then we speak about the univariate Hermite polynomials, which are usually denoted as $h(\xi)$, while if $\boldsymbol{\xi} = \{\xi_1, \xi_2, ..\xi_n\}$ is a vector of random variables, then we speak about the multivariate Hermite polynomials, which are denoted with $H(\boldsymbol{\xi})$. Multivariate Hermite polynomials are defined as a product of univariate Hermite polynomials

$$H_\alpha = \prod_{k=1}^{p} h_{\alpha_k}(\xi_k) \tag{4.5.19}$$

where p is a number of a random variables, while $\alpha = \{\alpha_1, \alpha_2, ...\alpha_p\}, \alpha_j \geq 0$ is a sequence of non-negative integers called multi-indices.
Example:

If the random vector is $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ and multi-index is given with the $\alpha = \{1, 0, 1, 0\}$, then multivariate Hermite polynomial is given as a product

$$H_\alpha = h_{\alpha_1}(\xi_1)h_{\alpha_2}(\xi_2)h_{\alpha_3}(\xi_3)h_{\alpha_4}(\xi_4) \tag{4.5.20}$$

i.e.

$$H_\alpha = h_1(\xi_1)h_0(\xi_2)h_1(\xi_3)h_0(\xi_4) = \xi_1 \cdot 1 \cdot \xi_3 \cdot 1 = \xi_1\xi_3. \tag{4.5.21}$$

Multivariate Hermite polynomials have also orthogonality property

$$\mathbf{E}[H_\alpha, H_\beta] = \delta_{\alpha\beta} \prod_{i=1}^{p} \alpha_i!, \tag{4.5.22}$$

where

$$\| H_\alpha \|^2_{L^2} = \alpha!. \tag{4.5.23}$$

From now on under the word Hermite polynomial we consider the normalised Hermite polynomials.

4.6 The Hermite Algebra

Product of two polynomials in general is again a polynomial. The same situation is with Hermite polynomials. Every Hermite polynomial can be expanded as a linear combination of a Hermite polynomials. First, we will observe expansion of a product of two Hermite polynomials and then we will observe the expansion of a product of more then two of them [109].

Lemma *For any nonnegative integer α and β, denote $\alpha \wedge \beta = min\{\alpha, \beta\}$. We have*

$$h_\alpha(\xi)h_\beta(\xi) = \sum_{p\le\alpha\wedge\beta} B(\alpha,\beta,p)h_{\alpha+\beta-2p}(\xi), \tag{4.6.1}$$

where

$$B(\alpha,\beta,p) = [\begin{pmatrix} \alpha \\ p \end{pmatrix}\begin{pmatrix} \beta \\ p \end{pmatrix}\begin{pmatrix} \alpha+\beta-2p \\ \alpha-p \end{pmatrix}]^{1/2}. \tag{4.6.2}$$

Proof Finding a product of a two generalised function of a Hermite polynomials, we have

$$\phi(\xi,z)\phi(\xi,t) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\frac{P_\alpha(\xi)P_\beta(\xi)}{\alpha!\beta!}z^\alpha t^\beta, \tag{4.6.3}$$

or using the exponential definition

$$\begin{aligned}\phi(\xi,z)\phi(\xi,t) &= e^{-(z^2+t^2)/2+(z+t)\xi} = e^{zt}e^{\xi^2/2}e^{-(z+t-\xi)^2/2} = \\ &= \sum_{p=0}^{\infty}\frac{(zt)^p}{p!}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(\xi)}{k!}(z+t)^k \\ &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{\mathcal{H}_k(\xi)}{p!}\sum_{0\le m\le k}\frac{1}{k!}\begin{pmatrix} k \\ m \end{pmatrix}z^{m+p}t^{k+p-m}.\end{aligned} \tag{4.6.4}$$

Let $k = m+\nu$, then $m \le k$ is equivalent to $\nu \ge 0$. The above formula can be rewritten as

$$\phi(\xi,z)\phi(\xi,t) = \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\mathcal{H}_{m+\nu}(\xi)}{p!}z^{m+p}t^{\nu+p}. \tag{4.6.5}$$

Using new notation $m+p=\alpha, \nu+p=\beta$ and conditions $m=\alpha-p\ge 0$, $\nu=\beta-p\ge 0$ we get $p \le \alpha\wedge\beta$. So,

$$\phi(\xi,z)\phi(\xi,t) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}(\sum_{m+p=\alpha,\nu+p=\beta}\frac{P_{m+\nu}(\xi)}{p!m!\nu!})z^\alpha t^\beta = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{p\le\alpha\wedge\beta}\frac{P_{\alpha+\beta-2p}(\xi)}{p!(\alpha-p)!(\beta-p)!}z^\alpha t^\beta. \tag{4.6.6}$$

Comparing equations we will get [109]

$$\mathcal{H}_\alpha(\xi)\mathcal{H}_\beta(\xi) = \sum_{p\le\alpha\wedge\beta}\frac{\alpha!\beta!}{p!(\alpha-p)!(\beta-p)!}\mathcal{H}_{\alpha+\beta-2p}(\xi). \tag{4.6.7}$$

Plugging $\mathcal{H}_n(\xi) = (n!)^{1/2}h_n(\xi)$ gives us a formula in the lemma.

Like a conclusion product of two univariate Hermite polynomials is given as [120, 121]

$$h_k(\xi)h_l(\xi) = \sum_{n=|k-l|}^{k+l}c_{kl}^n h_n(\xi). \tag{4.6.8}$$

The coefficients are only non-zero for integer $g = (k + l + n)/2 \in N$ and if $g \geq k \wedge g \geq l \wedge g \geq n$. They can be explicitly given as

$$c_{kl}^{(n)} = \frac{k!l!}{(g-k)!(g-l)!(g-n)!}, \tag{4.6.9}$$

and are called the structure constants of the univariate Hermite algebra. In the case of the multivariate Hermite polynomials, analogous statement holds

$$H_\alpha(\boldsymbol{\xi})H_\beta(\boldsymbol{\xi}) = \sum_\gamma c_{\alpha\beta}^{(\gamma)} H_\gamma(\boldsymbol{\xi}), \tag{4.6.10}$$

with the multivariate structure constants:

$$c_{\alpha\beta}^{\gamma} = \prod_{j=1}^{\infty} c_{\alpha_j\beta_j}^{(\gamma_j)}, \tag{4.6.11}$$

defined in the terms of the univariate constants. In this case we can calculate the expectation of the three Hermite polynomials

$$\mathbf{E}(H_\alpha H_\beta H_\gamma) = \mathbf{E}(H_\gamma \sum_\delta c_{\alpha\beta}^{(\delta)} H_\delta = c_{\alpha\beta}^{(\gamma)} \gamma!. \tag{4.6.12}$$

Product of more then two Hermite polynomials can be computed recursevly using the product of two Hermite polynomials.

4.7 Polynomial Chaos Expansion

Fundamental concept on which polynomial chaos (PC) decomposition is based is to regard uncertainty as generating new dimension and the solution as being dependent on this dimension [179]. A convergent expansion along new dimension is then sought in terms of a set of orthogonal basis functions, whose coefficients quantify and characterise the uncertainty.

The polynomial chaos is a particular basis of the space of random variables $L^2(\Omega, \mathcal{B}, P)$ based on Hermite polynomials of standard normal variables. Set of all polynomials $H_{i,\alpha}(\omega)$ with all possible sequences $(i; \alpha)$ of any length p forms a basis in $L^2(\Omega, \mathcal{B}, P)$.

Now let us denote by Γ_p subspace of $L^2(\Omega, \mathcal{B}, P)$ which is a homogeneous chaos of order p and its defined as space [59]:

$$\Gamma_p(\xi_1(\omega), ..., \xi_p(\omega)) = \{H_{i,\alpha}(\omega) \quad | \sum_{k=1}^{p} \alpha = p\}. \tag{4.7.1}$$

These subspaces are mutually orthogonal. It is proved by the fact that they are spanned by sets having null intersection.

Finally, the orthogonal decomposition [76] is:

$$L^2 = \bigoplus_{k=0}^{\infty} \Gamma_k \tag{4.7.2}$$

where $\oplus$ denotes the operator of orthogonal summation of subspaces in linear algebra. This decomposition is known as the Wiener Chaos decomposition. The random variable $\kappa(\omega)$ has a decomposition of the shape [59]:

$$\kappa(\omega) = \kappa_0\Gamma_0 + \sum_{i_1=0}^{\infty} \kappa_{i_1}\Gamma_1(\xi_{i1}(\omega)) + \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \kappa_{i_1 i_2}\Gamma_2(\xi_{i2}(\omega)) + ... \tag{4.7.3}$$

for each displacement degree of freedom where Γ_p are polynomial chaoses of order p of the arguments. The explicit forms of Γ_p up to the second order are:

$$\Gamma_0 = 1, \Gamma_1(\xi_i) = \xi_i, \Gamma_2(\xi_i) = \xi_i\xi_j - \delta_{i,j} \tag{4.7.4}$$

where δ_{ij} is the Kronecker delta. Using this equation, equation (4.7.3) can be rewritten in a following way [185]:

$$\kappa(\omega) = \sum_{i_1=0}^{\infty} \kappa^j H_j(\boldsymbol{\xi}(\omega)). \tag{4.7.5}$$

There is an one-to-one correspondence among the functions $\Gamma_p(\xi_1(\omega), ..., \xi_p(\omega))$ and $H_j(\boldsymbol{\xi}(\omega))$ and their corresponding coefficients. Again $H_j(\boldsymbol{\xi}(\omega))$ are the (multidimensional) orthogonal polynomials in terms of the multidimensional random vector $\boldsymbol{\xi}$, satisfying the orthogonality relation:

$$\langle H_i, H_j \rangle = \langle H_i^2 \delta_{ij} \rangle \tag{4.7.6}$$

where δ_{ij} is the Kronecker delta and $\langle ., . \rangle$ denotes the ensemble average.

	Random variables ξ	Wiener-Askey Chaos $\Phi(\xi)$	Support
Continuous	Gaussian	Hermite Chaos	$\{-\infty, \infty\}$
	Gamma	Laguerre Chaos	$[0, \infty\}$
	Beta	Jacobi Chaos	$[a, b]$
	Uniform	Legendre Chaos	$[a, b]$
Discrete	Poisson	Charlier Chaos	$\{0, 1, 2, ...\}$
	Binomial	Krawtchouk Chaos	$\{0, 1, ..., N\}$
	Negative Binomial	Meixner Chaos	$\{0, 1, 2, ..\}$
	Hypergeometric	Hahn Chaos	$\{0, 1, ..., N\}$

Table 1. *Wiener-Askey Chaos table*

In there is an one-to-one correspondence between the type of the orthogonal polynomials H and the type of the random variables $\boldsymbol{\xi}$. This is determined by choosing the type of the orthogonal polynomials H in such a way that their weighting function $w(\xi)$ in the

orthogonality relation of (4.7.6) has the same form as the probability distribution function of the underlying random variables $\boldsymbol{\xi}$. For example, the weighting function of n-dimensional Hermite orthogonal polynomials is the same as the probability density function of the n dimensional Gaussian random variables $\boldsymbol{\xi}$. Hence, the classical Wiener polynomial chaos is an expansion of Hermite polynomials in terms of Gaussian random variables. Some types of generalised polynomial chaos corresponding to the commonly known distributions are listed in Table 1.

The expansion resides in an infinite-dimensional space determined by $\boldsymbol{\xi}$ is an infinite summation. In practice, we have to restrict ourselves to the finite-term summation. This is achieved by reducing the expansion to a finite-dimensional space, i.e. expansion of finite-dimensional random variables $\boldsymbol{\xi}$, according to the nature of random inputs; we also set the highest order of the polynomials H according to accuracy requirement. The finite-term expansion takes the form [33, 185]:

$$\kappa(\boldsymbol{\xi}) = \sum_{i_1=0}^{M} \kappa^j H_j(\boldsymbol{\xi}) \tag{4.7.7}$$

where $\boldsymbol{\xi}$ is an n-dimensional random vector.

Shortly, the PC expansion of random variable of finite variance based on Hermite polynomials is the L^2-convergent expansion given by

$$f = \sum_{\alpha} f^{\alpha} H_{\alpha}, \tag{4.7.8}$$

where coefficients of expansion [48] are given as

$$f^{\alpha} = \frac{\mathbf{E}(f H_{\alpha})}{\| H_{\alpha} \|_{L^2}^2} \tag{4.7.9}$$

or in notation

$$f^{\alpha} = \frac{\langle f H_{\alpha} \rangle}{H_{\alpha}^2}. \tag{4.7.10}$$

In the case of smooth random variables coefficients may be computed analytically. If all its partial derivatives belong to L^2 then

$$f^{\alpha} = \alpha! \mathbf{E}(D^{\alpha} f), \tag{4.7.11}$$

where D^{α} is the partial derivative with respect to the multi-index. In previous equations summation is given by the index α representing the number of terms in summation. That number depends of a number of random variables M, which are independent Gaussian random variables, and order of the Hermite polynomials which we want to use i.e. order of polynomial chaos p. The total number of terms is given by

$$\begin{pmatrix} M+p \\ M \end{pmatrix}. \tag{4.7.12}$$

Practically polynomial chaos is given by array of coefficients and array of corresponding multivariate Hermite polynomials.

Example:

Let number of random variables in polynomial chaos be $M = 4$ and the order of a polynomial chaos is $p = 2$, then we have the polynomial basis given by the multi-index.

Multi-index	Polynomial basis
0000	$H_0 = 1$
1000	$H_1 = h_1(\xi_1) = \xi_1$
0100	$H_2 = h_1(\xi_2) = \xi_2$
0010	$H_3 = h_1(\xi_3) = \xi_3$
0001	$H_4 = h_1(\xi_4) = \xi_4$
2000	$H_5 = h_2(\xi_1) = \xi_1^2 - 1$
1100	$H_6 = h_1(\xi_1)h_1(\xi_2) = \xi_1\xi_2$
0200	$H_7 = h_2(\xi_2) = \xi_2^2 - 1$
1010	$H_8 = h_1(\xi_1)h_1(\xi_3) = \xi_1\xi_3$
0110	$H_9 = h_1(\xi_2)h_1(\xi_3) = \xi_2\xi_3$
0020	$H_{10} = h_2(\xi_3) = \xi_3^2 - 1$
1001	$H_{11} = h_1(\xi_1)h_1(\xi_4) = \xi_1\xi_4$
0101	$H_{12} = h_1(\xi_2)h_1(\xi_4) = \xi_2\xi_4$
0011	$H_{13} = h_1(\xi_3)h_1(\xi_4) = \xi_3\xi_4$
0002	$H_{14} = h_2(\xi_4) = \xi_4^2 - 1$

Table 2. Hermite polynomials for PC expansion of order 2

As we can notice from the above table, multi-index starts with zeros, representing H_0 polynomial and then it is built in second part in a way that sum of all indexes is 1, then in third horizontal part of a table in a way so that sum is equal 2. Conclusion is that sequence of multi-indexes is formed so that sum of all indices in one sequence is less or equal to the order of polynomial chaos p.

4.8 Polynomial Chaos Algebra

The most basic operations on PC variables are additions and subtractions, which are performed by adding/subtracting the corresponding PC coefficients of the variables being added/subtracted [33]. Multiplications of PC variables, however, are a little less straightforward.

Let observe two polynomial chaos expansion A and B given as

$$A = \sum_{j=0}^{p} A_j H_j, \quad \sum_{j=0}^{p} B_j H_j, \tag{4.8.1}$$

which are also called polynomial chaos variables or PC variables, then summation of these two PC variables gives us a new PC variable C

$$C = \sum_{j=0}^{p} C_j H_j = A + B = \sum_{j=0}^{p} A_j H_j + \sum_{j=0}^{p} B_j H_j = \sum_{j=0}^{p} (A_j + B_j) H_j \tag{4.8.2}$$

with coefficients

$$C_j = A_j + B_j. \tag{4.8.3}$$

In a similar way it is given subtraction of two PC variables

$$C = A - B, \quad C_j = A_j - B_j. \tag{4.8.4}$$

Algebraic operations of adding and subtracting are quite simple and basic, while operations of multiplying and dividing are more complicated. The product of two PC variables is given by new PC variable C

$$C = \sum_{k=0}^{p} C_k H_k = A \cdot B = \sum_{i=0}^{p} A_i H_i \cdot \sum_{j=0}^{p} B_j H_j, \tag{4.8.5}$$

where coefficients are given by

$$C_k = \sum_{i=0}^{p} \sum_{j=0}^{p} c_{ijk} A_i B_j, \quad \forall k \in 0, 1, ..., p, \tag{4.8.6}$$

where

$$c_{ijk} = \frac{\langle H_i H_j H_k \rangle}{\langle H_k^2 \rangle}. \tag{4.8.7}$$

Since the third order tensor c_{ijk} is only a function of the Hermite polynomials, it needs to be calculated only once during a preprocessing step and can be stored for use throughout the computations. Another advantage is the fact that tensor is sparse, reducing the amount of storage and needed CPU time. The similar procedure determines PC expansion of the product of three PC variables. Instead of direct calculation, in practice is used product of two PC random variables multiplied by the third one. This method is called pseudo-spectral method, not requiring the evaluation and storage of the 4-th order tensor [33].

Notation *It is very important to notice that product of two random variables is expressed using the polynomial chaos expansion of the same order as random variables itself, even though the right hand side of the product has a twice the order. The coefficients of product are therefore obtained by Galerkin projection minimizing the error of the resulting PC representation within the space spanned by the basis functions up to order p. Also, it is important to emphasise that both in the single but especially the repeated multiplications, the Galerkin projections onto a $p+1$ term PC expansion are essentially truncations, which introduce additional approximation errors. It is shown in [33] that those truncation errors are negligible if the order of the PC expansions is chosen sufficiently high to represent properly the result of the multiplication.*

Division of two PC variables is based on the problem of multiplication where the one of the factors in the product is unknown. For example let consider division of PC variables C and B. Then the problem is the same as the problem of solving equation

$$A = C/B \Rightarrow A \cdot B = C, \tag{4.8.8}$$

with unknown A. Having

$$\begin{aligned} C_1 &= \sum_{i=0}^{p}\sum_{j=0}^{p} c_{ij1}A_iB_j = \\ &= c_{001}A_0B_0 + c_{011}A_0B_1 + ... + c_{0p1}A_0B_p + ... + c_{p01}A_pB_0 + c_{p11}A_pB_1 + ... + c_{pp1}A_pB_p \end{aligned} \tag{4.8.9}$$

previous equation becomes a system of $p+1$ linear equations in the unknown coefficients $A_i, i = \{0, ..., p\}$

$$\begin{aligned} C_0 &= (c_{000}B_0 + c_{010}B_1 + ... + c_{0p0}B_p)A_0 + ... + (c_{p00}B_0 + c_{p10}B_1 + ... + c_{pp0}B_p)A_p \\ C_1 &= (c_{001}B_0 + c_{011}B_1 + ... + c_{0p1}B_p)A_0 + ... + (c_{p01}B_0 + c_{p11}B_1 + ... + c_{pp1}B_p)A_p \\ C_2 &= (c_{002}B_0 + c_{012}B_1 + ... + c_{0p2}B_p)A_0 + ... + (c_{p02}B_0 + c_{p12}B_1 + ... + c_{pp2}B_p)A_p \\ &... \\ C_p &= (c_{00p}B_0 + c_{01p}B_1 + ... + c_{0pp}B_p)A_0 + ... + (c_{p0p}B_0 + c_{p1p}B_1 + ... + c_{ppp}B_p)A_p. \end{aligned} \tag{4.8.10}$$

For solving this system can be used any classical numerical method for solving system of equations.

Using the method of division two PC variables one can calculate the inverse of the PC variable. Instead of C in previous equations we will have just C_0 which is equal to 1 and all other coefficients equal to zero. The problem becomes again problem of solving system of linear equations, well known in a literature and practice.

4.9 Non-polynomial Functions of a PC variables

All basic PC variable operations are quite easy to obtain. More difficulty bring us the non-polynomial functions like exponential or logarithm. Non-polynomial function evaluations of PC variables make a challenge since the Galerkin projection method cannot be applied directly to determine the PC coefficients of the function result. Several approaches can be used like Taylor expansion method, then Monte Carlo method, as well as integration approach [33].

The problems arising in this case are inaccuracy or cost-effectivity of the method. For example, Taylor expansion method is generally cost-effective and its accuracy depends of the number of terms in expansion. The more terms we use the accuracy is higher, but CPU time is larger. Under some conditions this approach doesn't converge at all. Usually, Taylor expansion method is not used for this kind of problems.

The general form of the problem is to calculate the function $f(A)$, where A is given as $A = \sum_{j=0}^{p} A_j H_j$ i.e it is polynomial chaos expansion.

4.9.1 Taylor Series Approach

For calculating non-polynomial function of a PC can be used Taylor series approach, based on expansion of a function in Taylor series around the mean value of PC variable A_0

$$f(A) = f(A_0) + f'(A_0)(A - A_0) + \frac{f''}{2!}(A - A_0)^2 + ... \tag{4.9.1}$$

Using new PC variable $B = A - A_0 = \sum_{j=1}^{p} A_j H_j$ and truncating the expansion, one gets m-th order approximation

$$f(A) = f(A_0) + f'(A_0)B + \frac{f''}{2!}B^2 + ... + \frac{f^{(m)}}{m!}B^m. \tag{4.9.2}$$

`Example:`
`Exponential function of a PC variable is given as`

$$e^A = e^{A_0} + e^{A_0}(A - A_0) + ... + \frac{e^{A_0}}{m!}(A - A_0)^m. \tag{4.9.3}$$

The Taylor series approach works as well as long the probability density functions are not too skewed, however high order PC variables are required to capture this stochastic information and the evaluation of high power terms in the Taylor expansion can become inaccurate.

4.9.2 Laguerre Expansion

Every function can be expressed in therms of Laguerre polynomials. The Laguerre polynomials can be expressed by the Rodrigues formula

$$L_n(x) = \frac{1}{n! e^{-x}} \frac{d^n}{dx^n}(x^n e^{-x}); \quad n = 0, 1, 2, 3, ... \tag{4.9.4}$$

They complete orthogonal set on the interval $0 \leq x \leq \infty$ with respect to the weighting function e^{-x}. Orthogonality is expressed by relation

$$\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \delta_{mn}. \tag{4.9.5}$$

By using this orthonality piecewise continuous function of a polynomial chaos expansion $f(x)$, where $x = \sum_{j=0}^{\infty} x_j H_j$, can be expressed in terms of Laguerre polynomials

$$f(x) = \sum_{n=0}^{\infty} C_n L_n(x), \tag{4.9.6}$$

where

$$C_n = \int_0^\infty e^{-x} f(x) L_n(x) dx. \tag{4.9.7}$$

Recurrent relation for Laguerre polynomials is given

$$L_n(x) = \frac{(2n-1)}{n} L_{n-1}(x) - \frac{x}{n} L_{n-1}(x) - \frac{(n-1)}{n} L_{n-2}(x), \tag{4.9.8}$$

and can be directly used for calculation of the function, where first Laguerre polynomial is $L_1 = 1$.

Example:

Let consider the square root function $f(x) = \sqrt{x}$, then coefficients of Laguerre expansion are given as

$$C_0 = \int_0^\infty e^{-x} x^{1/2} L_0(x) dx = \int_0^\infty e^{-x} x^{1/2} dx = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}, \tag{4.9.9}$$

where Γ is the gamma function, which interpolates the factorial.For the second coefficients we have

$$C_1 = \int_0^\infty e^{-x} x^{1/2} (1-x) dx = \Gamma(\frac{3}{2}) - \Gamma(\frac{5}{2}) = -\frac{\sqrt{\pi}}{4}, \tag{4.9.10}$$

and so on.

Laguerre polynomials	L_0	L_1	L_2	L_3	L_4	L_5	L_6
Coefficients	$\frac{\sqrt{\pi}}{2}$	$\frac{-\sqrt{\pi}}{4}$	$\frac{\sqrt{\pi}}{16}$	$\frac{-\sqrt{\pi}}{32}$	$\frac{-5\sqrt{\pi}}{2}$	$\frac{-7\sqrt{\pi}}{512}$	$\frac{-21\sqrt{\pi}}{2048}$

Notation: We need to be careful if we want to use this expansion for the function of PC variable. Because of the definition of the x in the previous integral, we need to divide polynomial chaos expansion with the mean value, and when we find the expansion using such normalised x, then in the end we have to multiply the square root with the $\sqrt{x_0}$.

4.9.3 Chebyshev Expansion

Chebyshev polynomials can be expressed using the Rodrigues formula

$$T_n(x) = \frac{\sqrt{1-x^2}}{(-1)^n (2n-1)!!} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}, \quad n = 0, 1, 2, 3, ... \tag{4.9.11}$$

They complete orthogonal set on the interval $-1 \leq x \leq 1$ with respect to the weighting function $\frac{1}{\sqrt{1-x^2}}$. Orthogonality is expressed by relation

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x) dx = p, \tag{4.9.12}$$

where $p = 0$ for $m \neq n$, $p = \pi$ for $m = n = 0$ and $p = \pi/2$ for $m = n = 1, 2, 3, ...$ By using this orthogonality piecewise continuous function $f(x)$ in $-1 \leq x \leq 1$ can be expressed in therms of Chebyshev polynomials

$$f(x) = \sum_{n=0}^{\infty} C_n T_n(x), \tag{4.9.13}$$

where

$$C_n = \frac{i}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) T_n(x) dx, n = 0, 1, 2, 3, .. \tag{4.9.14}$$

where $i = 1$ for $n = 0$ and $i = 2$ for $n = 1, 2, 3, ...$ Recurrence relation for Chebyshev polynomials is given

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x) \tag{4.9.15}$$

where first polynomial is equal $T_0(x) = 1$.
Example:
Let consider PC variable given as $x = \sum_{j=0}^{p} x_j H_j$ and divide it with the mean value, then we have $x^* = 1 + \sum_{j=1}^{p} \frac{x_j}{x_0} H_j = 1 + \sum_{j=1}^{p} x_j^* H_j$. Then Chebyshev expansion of a function $\frac{f(x)}{\sqrt{x_0}} = \frac{\sqrt{x}}{\sqrt{x_0}} = \sqrt{1 + x^*}$ is given by the coefficients

$$C_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^{*2}}} \sqrt{1+x^*} T_n^* dx^* = -2\sqrt{2}/\pi, \tag{4.9.16}$$

and so on with other coefficients.

4.9.4 Pade Approximation

All methods we described were rational function approach for solving square root problem. Pade approximation is a method based on Maclaurin series expansion (coefficient of it). The goal is to find the rational function described with numerator and denumerator, which can be used instead of Maclaurin series. Pade approximation is given by a rational fraction

$$[L/M] = \frac{a_0 + a_1 x + ... + a_L x^L}{b_0 + b_1 x + ... + b_M x^M} \tag{4.9.17}$$

which must agree with the Maclaurin expansion. Usually $b_0 = 1$, so there are $L + 1$ independent numerator coefficients and M independent denuminator coefficients.

$$\sum_{i=0}^{\infty} c_i x^i = \frac{a_0 + a_1 x + ... + a_L x^L}{b_0 + b_1 x + ... + b_M x^M} + O(x^{L+M+1}), \tag{4.9.18}$$

i.e.

$$(b_0 + b_1 x + ... + b_M x^M)(c_0 + c_1 x + ...) = a_0 + a_1 x + ... + a_L x^L + O. \tag{4.9.19}$$

Equating the coefficients, we get the system of m equations

$$\begin{aligned} &b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + ... + b_0 c_{L+1} = 0 \\ &b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + ... + b_0 c_{L+2} = 0 \\ &... \\ &b_M c_L + b_{M-1} c_{L+1} + ... + b_0 c_{L+M} = 0 \end{aligned} \tag{4.9.20}$$

Solving this system we have

$$Q^{[L/M]}(x) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} & c_{L+2} \\ & & \cdots & & \\ c_{L-1} & c_L & \cdots & c_{L+M-2} & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M-1} & c_{L+M} \\ x^M & x^{M-1} & \cdots & x & 1 \end{vmatrix} \tag{4.9.21}$$

$$P^{[L/M]}(x) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} \\ & & \cdots & \\ c_{L-1} & c_L & \cdots & c_{L+M-2} \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \\ \sum_{i=0}^{L-M} c_i x^{M+i} & \sum_{i=0}^{L-M+1} c_i x^{M+i-1} & \cdots & \sum_{i=0}^{L} c_i x^i \end{vmatrix} \tag{4.9.22}$$

so that

$$Q^{[L/M]}(x) \sum_{i=0}^{\infty} c_i x^i - P^{[L/M]}(x) = O(x^{l+M+1}). \tag{4.9.23}$$

Pade approximant is given with

$$[L/M] = \frac{P^{[L/M]}(x)}{Q^{[L/M]}(x)}, \tag{4.9.24}$$

with condition $Q^{[L/M]}(x) \neq 0$.

4.9.5 Newton Method for Square Root

The Newton method based on a Taylor approximation up to first order is the most used method for finding square root. When it deals with numbers, one has that this method is efficient and accurate. The negative property is that its convergence depends a lot of initial point i.e. if initial point is close to solution, it will give us exact solution, otherwise one can get wrong solution or method will not converge. When we speak about PC variables, the method will be the same as for a numbers, just replacing number with the PC variable and algebraic operations of adding and dividing with the polynomial chaos

algebraic operations of adding and dividing. The problem here arise when we speak about division. This operation is based on solving system of linear algebraic equations. Therefore, in each iteration in every division one spend time on solving this system and method of finding square root becomes unefficient. To overcome this, we observe process of finding the root of inverse function instead. In that way we will have just one division per iteration, making method more efficient.
Let consider PC variable $x = \sum_{j=0}^{p} x_j H_j$, then the square root given by Newton method is:

$$u_{n+1} = \frac{1}{2}(u_n + \frac{x}{u_n}), \tag{4.9.25}$$

where iterations repeat until the values of u_n and x be the same or in some tolerance defined with

$$\text{tol} = \sqrt{x_0^2 + \text{var}}, \tag{4.9.26}$$

where x_0 is a mean value of PC and var is a variance of it.

The problem is still initial point i.e. where to start to look solution to get convergent method. If we are too far away of our solution, method will be pretty slow or will not converge.

Notation *The initial point must be also PC variable and often one can use a mean value of a PC variable i.e. first coefficient will be equal to mean value other ones will be equal to zero.*

4.10 Conclusion

Random fields are described as the region where we assign a random variable to each point of that region. The Karhunen-Loève expansion furnish desire for decomposition of random fields similar to a spectral synthesis, given in introduction of this chapter. It decompose random fields into the products of deterministic functions on the region and functions dependent on ω i.e. simple random variables. If random field is Gaussian then random variables are also Gaussian, as also their linear combination. These random variables are uncorrelated and independent making synthesis very simple. Non-Gaussian fields are little bit more complicated in a way that one need to synthesise them as a functions of Gaussian fields. These functions are usually nonlinear transformations. For known covariance function one can consider Fredholm equation i.e. eigenproblem, which cannot be solved analytically, but using standard numerical methods. Reyleigh-Ritz method is often used to discretise Karhunen-Loève eigenvalue problem by projecting onto finite-dimensional subspace.

The main problem is that covariance function of the solution is not known a priori, and hence the KL expansion may not be used to approximate it. Although, Karhunen-Loève representation may not be suitable approximation even when much is known about

the covariance function of the solution. Sometimes we have a problem with Gaussian parameters which give us not necessarily Gaussian solution. That problem is solved using new means representation called polynomial chaos expansion. Polynomial chaos expansion represents any random variable as a series of polynomials in uncorrelated and independent Gaussian variables.

If polynomial chaos expansion is observed as the new kind of variable, then one can define polynomial chaos algebra. Some operations are very easy to obtain, like adding, while another are more complicated like dividing. In that case some standard numerical methods are used. More complicated problem is to calculate non-polynomial functions of these polynomial chaos variables. This is still area of exploring.

In the end, in this chapter is given the idea of random variables as functions in an infinite dimensional space, approximated by elements of finite dimensional spaces. This idea brings us Galerkin methods, becoming very useful in the approximation of partial differential and integral equations. With this comes the whole range of techniques for fast and efficient solution. In other words a stochastic problem becomes a large deterministic one.

Chapter 5

Basics of Plasticity

In this chapter we consider the basic formulation of the elastoplastic model beginning with equilibrium equation. The main goal is to describe the elastic perfectly plastic materials which are subject of our observation [40, 42, 173, 49].

5.1 General Theory of Elastoplasticity

The theory of elastoplastic material is based on the theory of thermodynamics [50, 117, 118, 123]. For further explanation we will consider Helmholtz free energy Ψ and stress $\boldsymbol{\sigma}$ [69], given as the functions of the strain $\boldsymbol{\varepsilon}$ and a set of m internal variables $\vartheta_1, \vartheta_2, .., \vartheta_m$. Some of these may be scalars and some tensors, depending of the application.

The constitutive equations are thus of the form:

$$\Psi = \Psi(\boldsymbol{\varepsilon}, \vartheta_1, \vartheta_2, .., \vartheta_m) \tag{5.1.1}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \vartheta_1, \vartheta_2, .., \vartheta_m). \tag{5.1.2}$$

Unlike the case of elasticity historical effects are relevant such that above equations are not sufficient and it is necessary to add to this pair of equations an evolution equation in which the rate of change of each internal variable ϑ_i is given by an equation of the form:

$$\dot{\vartheta}_i = \beta_i(\boldsymbol{\varepsilon}, \vartheta_1, \vartheta_2, .., \vartheta_m), \quad 1 \leq i \leq m. \tag{5.1.3}$$

Assume that all functions appearing are sufficiently smooth with respect to their arguments that as many derivatives as required may be taken. Then using the reduced dissipation inequality [69], the stress is given by:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \tag{5.1.4}$$

as well as internal forces χ_i, conjugate to ϑ_i

$$\chi_i = -\frac{\partial \Psi}{\partial \vartheta_i}, \quad 1 \leq i \leq m. \tag{5.1.5}$$

The dissipation inequality then becomes

$$\chi_i \colon \dot{\vartheta}_i \geq 0, \tag{5.1.6}$$

or a scalar product of force-like variables (χ_i) with the rate of change of strain-like variables (ϑ_i), so that dissipation rate due to the internal factors is nonnegative.

The main goal in plasticity theory is that for given state of stress and history of behaviour of the material point, one needs to express the strain role as a function of stress and the history. We will deal with two kinds of variables: kinematic quantities and dynamic quantities [42, 164]. Kinematic quantities are called also strain-like variables, while dynamic are stress-like variables. The kinematic variable which characterises local deformation is called a total strain $\boldsymbol{\varepsilon}$. For small deformations it can be decomposed into the elastic strain $\boldsymbol{\varepsilon}_e$, due to the elastic behaviour of material point and plastic strain $\boldsymbol{\varepsilon}_p$, which characterises the irreversible part of the deformation. The elastic and plastic strain, like the total strain are symmetric second-order tensors. Also, in this group enters a set of internal variables ϑ, which characterise hardening. The stress-like variables are the stress $\boldsymbol{\sigma}$ and a set $\boldsymbol{\chi}$ of internal forces, due to the internal restructuring that occurs during plastic deformation. Internal forces are conjugate with internal variables in the same way in which the stress is conjugate to the strain.

Now we can introduce generalised stress as a set $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ and generalised plastic strain $\mathbf{E}_p = (\boldsymbol{\varepsilon}_p, \boldsymbol{\vartheta})$. $\boldsymbol{\Sigma}$ and $\mathbf{E}_p$ are conjugate in the sense that product $\boldsymbol{\Sigma} : \dot{\mathbf{E}}_p = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p + \boldsymbol{\chi} : \dot{\boldsymbol{\vartheta}}$ represents the rate of dissipation due to the plastic deformation [69].

For definition of a strain one use Gibbs free energy Ψ_G

$$\Psi_G(\boldsymbol{\sigma}, \boldsymbol{\vartheta}) = \boldsymbol{\sigma} : \boldsymbol{\vartheta} - \Psi, \tag{5.1.7}$$

so that the total strain is given in the form

$$\boldsymbol{\varepsilon} = \frac{\partial \Psi_G}{\partial \boldsymbol{\sigma}}. \tag{5.1.8}$$

Both, the Helmholtz and Gibbs free energy can be decomposed into elastic and plastic part

$$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\vartheta}) = \Psi_e(\boldsymbol{\varepsilon}_e) + \Psi_p(\boldsymbol{\vartheta}) \equiv \hat{\Psi}(\boldsymbol{\varepsilon}_e, \vartheta), \tag{5.1.9}$$

$$\Psi_G(\boldsymbol{\sigma}, \boldsymbol{\vartheta}) = \Psi_{G_e}(\boldsymbol{\sigma}) + \Psi_{G_p}(\boldsymbol{\vartheta}), \tag{5.1.10}$$

where

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}_p(\boldsymbol{\vartheta}). \tag{5.1.11}$$

From previous equations one can conclude that the elastic strain in general case is only dependent of stress, while the plastic strain of internal variables.

At all times the generalised stress $\boldsymbol{\Sigma}$ lies in a closed, connected set $\mathcal{K}$ of admissible generalised stresses. The interior of this set is called the elastic region and is denoted by int $\mathcal{K}$. The boundary of $\mathcal{K}$ is denoted with $\partial\mathcal{K}$ and represents the yield surface.

Purely elastic behaviour takes place when $\boldsymbol{\Sigma} \in$ int $\mathcal{K}$ or when generalised stress moves from $\partial\mathcal{K}$ to the interior of $\mathcal{K}$ (elastic unloading). Plastic behaviour takes place only if $\boldsymbol{\Sigma}$

lies on the yield surface and continuous to lie on the yield surface (plastic loading). In other words, using function ϕ one can define yield surface $\partial\mathcal{K} = \{\boldsymbol{\Sigma} : \phi(\boldsymbol{\Sigma}) = 0\}$ and int $\mathcal{K} = \{\boldsymbol{\Sigma} : \phi(\boldsymbol{\Sigma}) < 0\}$, so that the rate of change of generalised plastic strain $\dot{\mathbf{E}}_p = 0$ if $\phi(\boldsymbol{\Sigma}) < 0$ or $\phi(\boldsymbol{\Sigma}) = 0, \dot{\phi} < 0$, and it is non-zero only if $\phi(\boldsymbol{\Sigma}) = 0, \dot{\phi} = 0$. This condition is called consistency condition and it is given by requirement

$$\phi = \dot{\phi} = 0. \tag{5.1.12}$$

In practice we observe projection of a function ϕ on the space of stresses $\bar{\phi}(\boldsymbol{\sigma}) \equiv \phi(\boldsymbol{\sigma}, \boldsymbol{\chi})$ for fixed $\boldsymbol{\chi}$, instead of function ϕ.

For the case of perfectly plastic behaviour one can notice that the elastic region as well as the yield surface depends only on the stress, so that $\phi(\boldsymbol{\sigma}) = \bar{\phi}(\boldsymbol{\sigma}) = 0$. In other words plastic behaviour [164] takes place when

$$0 = \dot{\phi} = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}; \tag{5.1.13}$$

i.e. the stress moves around the yield surface during plastic deformation and the yield surface in the stress space remains unchanged by this behaviour. This behaviour is known as a neutral loading unlike the case when yield surface changes during the plastic deformation.

Very important postulate of the plasticity theory is postulate of maximum virtual work [69], which says that for given stress $\boldsymbol{\sigma}$ for which $\phi(\boldsymbol{\sigma}) = 0$ and a plastic strain rate $\dot{\boldsymbol{\varepsilon}}_p$ associated with $\boldsymbol{\sigma}$ states

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p \geq \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}_p \tag{5.1.14}$$

for all admissible stresses $\boldsymbol{\tau}$ (i.e. which satisfies $\bar{\phi} \leq 0$). From this postulate two very important consequences arise. First one is known as a normality law and second one is the convexity of the region int $\mathcal{K}$ i.e. of the region $\mathcal{K}$.

Normality law states that for smooth yield surface rate of a change of a plastic strain $\dot{\mathbf{E}}_p$ is normal to the tangent hyperplane at the point $\boldsymbol{\Sigma}$ to the yield surface $\partial\mathcal{K}$, while for the non-smooth yield-surface it lies in the cone of normals at point $\boldsymbol{\Sigma}$. Using normality law for smooth yield surface (in every point has a well-defined gradient), one can express the generalised plastic strain rate

$$\dot{\mathbf{E}}_p = \gamma \nabla \phi(\boldsymbol{\Sigma}) \tag{5.1.15}$$

where γ is a nonnegative scalar, called plastic multiplier. For perfect plastic material it has form of

$$\dot{\boldsymbol{\varepsilon}}_p = \gamma \frac{\partial \phi}{\partial \boldsymbol{\sigma}}. \tag{5.1.16}$$

The function ϕ and scalar γ are nor simultaneously zero, but defined using consistency conditions:

$$\gamma \geq 0, \quad \dot{\phi} \leq 0, \quad \gamma \dot{\phi} = 0 \tag{5.1.17}$$

5.2 Definition of Spaces in Elastoplastic Problem

The object of our observing is plasticity occupying a bounded Lipschitz domain $\mathcal{G} \subset \mathbb{R}^d$ with boundary Γ [69, 96, 164]. Let introduce the function spaces for the variables of interest. First let $I = [0, T]$ be the interval of time and let assume for sake of simplicity, homogeneous geometric boundary conditions for the displacement field and the following space of admissible displacements

$$\mathcal{U} = [H_0^1(\mathcal{G})]^3. \tag{5.2.1}$$

Denote by $(\boldsymbol{\sigma}, \mathbf{u}) : I \to S \times \mathcal{U}$ the stress and displacement evolution. At all times stress lies in a closed, non-empty, connected set $\mathcal{K}$. The loading function is given as $\boldsymbol{f} : I \to \mathcal{F}$, where $\mathcal{F}$ is a space of forces, in duality with $\mathcal{U}$. The space of plastic strain is given as

$$\mathcal{E}_p = \{\boldsymbol{\varepsilon}_p \in L^2(\mathcal{G}; \mathbb{R}^{d\times d}_{sym}) : \text{tr } \boldsymbol{\varepsilon}_p = 0 \quad \text{a.e. in } \mathcal{G}\}, \tag{5.2.2}$$

while space of internal variables is denoted with $\Theta \in L^2(\mathcal{G})^m$.

The elasticity tensor $\mathbf{C}$ is bounded and has measurable components , that is

$$C_{ijkl} \in L^\infty(\mathcal{G}) \tag{5.2.3}$$

and it is pointwise stable. [1] Compliance tensor or inverse of the elastic tensor $\mathbf{C}^{-1}$ has also the same symmetry properties as $\mathbf{C}$ and it is also pointwise stable.

In this section we give just definition of a few spaces. All other will be considered below.

5.3 Equilibrium Equations

Here will be given full set of equations, which are assumed to be posed on a bounded Lipschitz domain $\mathcal{G}$ with boundary Γ.

5.3.1 Basic Formulation of the Problem

The plasticity is described with constitutive equations given below. The goal is to solve them and get unknown displacement $\boldsymbol{u}$, the plastic strain $\boldsymbol{\varepsilon}_p$ and the internal hardening variables $\boldsymbol{\vartheta}$. All constitutive equations are gathered in the next equations [14, 42, 69, 164]:

- Equilibrium equation

$$\text{div}\boldsymbol{\sigma} + \boldsymbol{f} = 0 \tag{5.3.1}$$

[1] there exist a constant $C_0 > 0$ such that

$$C_{ijkl}(x)\varsigma_{ij}\varsigma kl \geq C_0|\varsigma|^2, \quad \forall \varsigma = (\varsigma_{ij}) \in M^3, a.e. in \mathcal{G} \tag{5.2.4}$$

- Strain-displacement mapping

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) \tag{5.3.2}$$

- Hook's law

$$\boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p) \tag{5.3.3}$$

,

$$\boldsymbol{\chi} = -\mathbf{H}\boldsymbol{\vartheta} \tag{5.3.4}$$

where $\mathbf{H}$ is hardening modulus.

- The flow law

$$\dot{\mathbf{E}}_p = (\dot{\boldsymbol{\varepsilon}}_p, \dot{\boldsymbol{\vartheta}}) \in N_K(\boldsymbol{\Sigma}), \tag{5.3.5}$$

where $N_K(\boldsymbol{\Sigma})$ denotes the normal cone to $\mathcal{K}$ at $\boldsymbol{\Sigma}$. This is equivalent to :

$$\boldsymbol{\Sigma} \in \partial \mathcal{D}(\dot{\boldsymbol{E}}_p), \tag{5.3.6}$$

where $\partial \mathcal{D}(\dot{\boldsymbol{E}}_p)$ denotes the subdifferential of $\mathcal{D}$ at $\dot{\boldsymbol{E}}_p$, defined by:

$$\mathcal{D}(q, \eta) \geq \mathcal{D}(\dot{\boldsymbol{\varepsilon}}_p, \dot{\boldsymbol{\vartheta}}) + \boldsymbol{\sigma} \colon (\boldsymbol{q} - \dot{\boldsymbol{\varepsilon}}_p) + \boldsymbol{\chi} \colon (\boldsymbol{\eta} - \dot{\boldsymbol{\vartheta}}), \quad \forall (\boldsymbol{q}, \boldsymbol{\eta}) \in \mathcal{K}_p. \tag{5.3.7}$$

where $\mathcal{D}$ is a dissipation function (nonnegative, convex, positively homogeneous and l.s.c.). It is a measure of the rate of irreversible or plastic work.

5.3.2 Primal Problem

A primal problem [22, 68, 69] can be derived from constitutive equations given in the previous section (5.3.7, 5.3.3, 5.3.4, as well as 5.3.1). The definition of the primal problem rely on a set of variables $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \boldsymbol{\vartheta})$, treaten as primary variables.

In order to give basic formulation of this problem, one needs to introduce space Z. The product space $Z = \mathcal{U} \times \mathcal{E}_p \times \Theta$ is a Hilbert space with the inner product:

$$(\boldsymbol{w}, \boldsymbol{z}) = (\boldsymbol{u}, \boldsymbol{v})_{\mathcal{U}} + (\boldsymbol{\varepsilon}_p, \boldsymbol{q})_{\mathcal{E}_p} + (\boldsymbol{\vartheta}, \boldsymbol{\eta})_{\Theta} \tag{5.3.8}$$

and norm:

$$\|\boldsymbol{z}\|_Z = (\boldsymbol{z}, \boldsymbol{z})_Z^{1/2}, \tag{5.3.9}$$

where $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \boldsymbol{\vartheta})$ and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\eta})$.

Let introduce next few forms over the space Z like bilinear form

$$a(\boldsymbol{w}, \boldsymbol{z}) = \int_{\mathcal{G}} [\mathbf{C}(\boldsymbol{\varepsilon} - \boldsymbol{q}) + \boldsymbol{\vartheta} \cdot \mathbf{H}\boldsymbol{\eta}] dx, \tag{5.3.10}$$

then the linear functional

$$\langle l(t), \boldsymbol{z} \rangle = \int_{\mathcal{G}} \boldsymbol{f}(t) \cdot \boldsymbol{v} dx, \tag{5.3.11}$$

and convex, positevly homogeneous, nonnegative and l.s.c functional:

$$j(\boldsymbol{z}) = \int_{\mathcal{G}} \mathcal{D}(\boldsymbol{q}, \boldsymbol{\eta}) dx. \tag{5.3.12}$$

The primal problem then is defined as: for given $l \in H^1(I; Z'), l(0) = 0$ find $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \boldsymbol{\vartheta}) : I \to Z, \boldsymbol{w}(0) = 0$, such that for almost all $t \in I, \dot{\boldsymbol{w}}(t) \in Z_p$ and

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) \geq \langle l(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t) \rangle, \quad \forall \boldsymbol{z} \in Z_p, \tag{5.3.13}$$

where space Z_p is defined as

$$Z_p = \{\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\eta}) \in Z : (\boldsymbol{q}, \boldsymbol{\eta}) \in K_p \text{ a.e. in } \mathcal{G}\} \tag{5.3.14}$$

and it is nonempty, closed, convex cone in Z, where $K_p = \text{dom}\mathcal{D}$.

If $\boldsymbol{w}$ is a classical solution, then it is solution to the primal problem. If it is *smooth* solution of the primal problem then it is solution of a classical problem.

More about the primal problem one can see text book of Reddy [69] and paper [68].

5.3.3 Dual Problem

The definition of a dual rely on a set of variables $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\chi})$, treaten as primary variables [67, 69]. The space of displacement is given with $\mathcal{U}$, while the stress space and the conjugate force space are given as:

$$S = \boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji}, \tau_{ij} \in L^2(\mathcal{G}), \tag{5.3.15}$$

$$\Xi = \boldsymbol{\mu} = (\mu_j) : \mu_j \in L^2(\mathcal{G}), j = 1, 2, ..., m, \tag{5.3.16}$$

respectively. Next, one can introduce space

$$\mathcal{T} = S \times \Xi \tag{5.3.17}$$

which is endowed with the inner products induced by the natural inner products on Σ and Ξ. Admissible generalised stresses are those that belong to the set $\mathcal{K}$ pointwise, so that we have convex set

$$\mathcal{P} = \boldsymbol{T} = (\tau, \mu) \in \mathcal{T} : (\tau, \mu) \in \mathcal{K} \quad \text{a.e. in} \quad \mathcal{G}. \tag{5.3.18}$$

Let introduce bilinear forms

$$\bar{a} : S \times S \to \mathbb{R}, \quad \bar{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_{\mathcal{G}} \boldsymbol{\sigma} : \mathbf{C}^{-1} \boldsymbol{\tau} dx, \tag{5.3.19}$$

$$b : \mathcal{U} \times S \to \mathbb{R}, \quad b(v, \tau) = -\int_{\mathcal{G}} \boldsymbol{\varepsilon}(v) : \boldsymbol{\tau} dx, \tag{5.3.20}$$

$$c : \Xi \times \Xi \to \mathbb{R}, \quad c(\boldsymbol{\chi}, \boldsymbol{\mu}) = \int_{\mathcal{G}} \boldsymbol{\chi} : \mathbf{H}^{-1} \boldsymbol{\mu} dx, \tag{5.3.21}$$

and

$$A : \mathcal{T} \times \mathcal{T} \to \mathbb{R}, A(\Sigma, T) = \bar{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + c(\boldsymbol{\chi}, \boldsymbol{\tau}), \tag{5.3.22}$$

for $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $\boldsymbol{T} = (\boldsymbol{\tau}, \boldsymbol{\mu})$.

Also, let introduce the linear functional

$$l(t) : \mathcal{U} \to \mathbb{R}, \langle l(t), \boldsymbol{v} \rangle = -\int_{\mathcal{G}} \boldsymbol{f}(t) \cdot \boldsymbol{v} dx. \tag{5.3.23}$$

The dual variational problem is then obtained through the standard procedure from the equilibrium equation

$$\text{div}\boldsymbol{\sigma} + \boldsymbol{f} = 0 \text{ in } \mathcal{G} \tag{5.3.24}$$

and flow law:

$$(\dot{\boldsymbol{\sigma}}_E - \dot{\boldsymbol{\sigma}}) : \mathbf{C}^{-1}(\boldsymbol{\tau} - \boldsymbol{\sigma}) - \dot{\boldsymbol{\chi}} : \mathbf{H}^{-1}(\boldsymbol{\mu} - \boldsymbol{\chi}) \leq 0, \quad \forall (\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathcal{K}, \tag{5.3.25}$$

where elastic stress rate is given as

$$\boldsymbol{\sigma}_E = \mathbf{C}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}). \tag{5.3.26}$$

In other words problem is for given $l \in H^1(0, T; V')$ with $l(0) = 0$ to find $(\boldsymbol{u}, \boldsymbol{\Sigma}) = (\boldsymbol{u}, \boldsymbol{\Sigma}, \boldsymbol{\chi}) : I \to \mathcal{U} \times \mathcal{P}$ with $\boldsymbol{u}(0) = 0, \boldsymbol{\Sigma}(0) = 0$ such that for almost all $t \in I$,

$$b(\boldsymbol{v}, \boldsymbol{\sigma}(t)) = \langle l(t), \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in V, \tag{5.3.27}$$

$$A(\dot{\boldsymbol{\Sigma}}(t), T - \boldsymbol{\Sigma}(t) + b(\dot{\boldsymbol{u}}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t)) \geq 0, \quad \forall \boldsymbol{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathcal{P}. \tag{5.3.28}$$

In the case when internal forces and internal variables are equal to zero (i.e case of elasticity) system become

$$\int_{\mathcal{G}} \boldsymbol{\sigma} : \mathbf{C}^{-1}\boldsymbol{\tau} dx - \int_{\mathcal{G}} \boldsymbol{\varepsilon}(u) : \boldsymbol{\tau} dx = 0, \quad \forall \boldsymbol{\tau} \in S, \tag{5.3.29}$$

and

$$-\int_{\mathcal{G}} \boldsymbol{\varepsilon}(v) : \boldsymbol{\sigma} dx = -\int_{\mathcal{G}} \boldsymbol{f} \cdot \boldsymbol{v} dx, \quad \forall \boldsymbol{v} \in [H_0^1(\mathcal{G})]^3. \tag{5.3.30}$$

5.4 Different Types of Hardening Models

In previous chapter the flow rule and the consistency conditions describe elasto-perfect plastic materials, but perfect plasticity is not the only kind of model. There are some more types which include hardening. The hardening can be isotropic or kinematic [164, 42, 14].

The essential difference between perfect plasticity and isotropic hardening plasticity lies in the fact that for perfect plasticity the closure of elastic range int $\mathcal{K}$ remains unchanged, whereas for the strain hardening model, int $\mathcal{K}$ expands with the amount of the plastic flow. In this model, the hardening is assumed to obey two conditions [164]:

- the hardening is isotropic in the sense that any state of loading, the centre of int $\mathcal{K}$ remains in the origin;
- the hardening is linear in the amount of plastic flow (i.e. linear in $|\dot{\boldsymbol{\varepsilon}}^p|$) and independent of its sign (i.e. sign$(\dot{\boldsymbol{\varepsilon}}^p)$).

First of two conditions is described in mathematical way as:

$$\phi(\boldsymbol{\sigma}, \vartheta) = |\boldsymbol{\sigma}| - [\boldsymbol{\sigma}_Y + \mathbf{H}\vartheta] \leq 0, \quad \vartheta \geq 0 \tag{5.4.1}$$

where $\mathbf{H}$ is the plastic modulus and variable $\alpha : I \to \mathbb{R}$ is nonnegative function of plastic flow, called internal hardening variable. In the case when $\mathbf{H} < 0$, one speaks about stress softening response. The second condition describes the simplest evolutionary equation for ϑ:

$$\dot{\vartheta} = |\dot{\boldsymbol{\varepsilon}}^p| = \gamma. \tag{5.4.2}$$

In many metals subjected to cyclic loading, it is experimentally observed that the centre of the yield surface experiences a motion in the direction of the plastic flow. This phenomenon is described by model of kinematic hardening and it's constructed by introducing an additional internal variable called back stress and denoted by $\boldsymbol{\sigma}_b$. The back stress defines location of the centre of the yield surface.

The yield function in a case of combined isotropic and kinematic hardening is described as:

$$\phi(\boldsymbol{\sigma}, \boldsymbol{\sigma}_b, \vartheta) = |\boldsymbol{\sigma} - \boldsymbol{\sigma}_b| - [\boldsymbol{\sigma}_Y + \mathbf{H}\vartheta] \leq 0 \tag{5.4.3}$$

where the back stress $\boldsymbol{\sigma_b}$ is defined by Ziegler's rule:

$$\dot{\boldsymbol{\sigma}}_b = \mathbf{H}_{\text{kin}}\dot{\boldsymbol{\varepsilon}}^p \equiv \gamma \mathbf{H}_{\text{kin}} \text{ sign}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b). \tag{5.4.4}$$

$\mathbf{H}_{\text{kin}}$ denote the kinematic hardening modulus, while flow rule is given by:

$$\dot{\boldsymbol{\varepsilon}}^p = \gamma \text{ sign}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b) \tag{5.4.5}$$

5.5 Radial Return Mapping Algorithm

Radial return map is a numerical method often used to approach the solution of the described plasticity model [42, 105]. In general, method consists in:

- the time-integration of the differential algebraic system, leading to an algebraic system;
- the formulation of the solution algorithm for the obtained algebraic system.

Let time interval $I = [0, T]$ be divided into N sub-intervals defined by the points $0 = t_0 < t_1 < ... < t_n < t_{n+1} .. < t_N < t_{N+1}$. If t_n is a generic time instant, then $\boldsymbol{\varepsilon}_n^{\text{dev}}$ is the deviatoric strain at time t_n, $\boldsymbol{\sigma}_n^{\text{dev}}$ is the deviatoric stress at the same time, and so on for all the problem variables. It is supposed that strain history path is known (strain driven problem), while unknown variables are stress, plastic strain and total strain.

5.5.1 Time Continuous Model

Radial return map can be explained on the associative von Mises plasticity model with kinematic and isotropic hardening in the realm of small deformations [6, 23, 105, 108]. The stress tensor as well as strain tensor are separated on deviatoric and spherical part:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} + p\mathbf{1}, \quad p = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) \tag{5.5.1}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{dev} + \frac{1}{3}\boldsymbol{\varepsilon}^{sph}\mathbf{1}, \quad \boldsymbol{\varepsilon}^{sph} = \mathrm{tr}(\boldsymbol{\varepsilon}). \tag{5.5.2}$$

The volumetric elastic relation connects pressure with the total volumetric strain $\boldsymbol{\varepsilon}^{sph}$, introducing K, the material bulk modulus:

$$p = K\boldsymbol{\varepsilon}^{sph}, \tag{5.5.3}$$

while deviatoric stress is given by deviatoric elastic relation with G, the material shear modulus:

$$\boldsymbol{\sigma}^{dev} = 2G[\boldsymbol{\varepsilon}^{dev} - \boldsymbol{\varepsilon}_p^{dev}] \tag{5.5.4}$$

where $\boldsymbol{\varepsilon}_p^{dev}$ is a deviatoric plastic strain.

As it is said, in the case of kinematic hardening the centre of the yield surface change the position, so one can introduce the relative stress $\mathbf{s}$ given in terms of the backstress $\boldsymbol{\sigma}_b$:

$$\mathbf{s} = \boldsymbol{\sigma}^{dev} - \boldsymbol{\sigma}_b. \tag{5.5.5}$$

The isotropic hardening is described with equation:

$$\sigma_y = \sigma_{y,0} + \mathbf{H}_{iso}\gamma \tag{5.5.6}$$

which is a linear isotropic hardening mechanism, governing the radius of the yield surface, where $\sigma_{y,0}$ is the initial yield stress. In the case of kinematic hardening, the backstress determines the shift of the yield function in the stress space:

$$\dot{\boldsymbol{\sigma}}_b = \mathbf{H}_{kin}\dot{\boldsymbol{\varepsilon}}_p^{dev} \tag{5.5.7}$$

where the deviatoric part of plastic strain is calculated as it is shown in previous chapter:

$$\dot{\boldsymbol{\varepsilon}}_p^{dev} = \dot{\gamma}\boldsymbol{n}. \tag{5.5.8}$$

Previous equation determines evaluation of the plastic strain, where $\boldsymbol{n}$ is defined as

$$\boldsymbol{n} = \frac{\partial\phi}{\partial\mathbf{s}} = \frac{\mathbf{s}}{\|\mathbf{s}\|}. \tag{5.5.9}$$

Although, always it must be satisfied yield function:

$$\phi = \|\mathbf{s}\| - \sigma_{y,0} \leq 0 \tag{5.5.10}$$

as well as Kuhn-Tucker conditions already introduced in previous chapter. It is important to notice that kinematic and isotropic hardening are linear functions.

5.5.2 Radial Return Map

Once more, at beginning of the step it is assumed to know the values ($\boldsymbol{\sigma}_n^{dev}$, $\boldsymbol{\varepsilon}_n^{dev}$, $\gamma_n, \boldsymbol{\sigma}_{bn}$) (γ_n is a consistency parameter, while $\boldsymbol{\sigma}_{bn}$ is a backstress) at time t_n, and the deviatoric strain $\boldsymbol{\varepsilon}_n^{dev}$ at time t_{n+1}. The goal is to find remaining variables at time t_{n+1} [6, 23, 108, 105]. Radial return map is a numerical method where first one calculate trial values, supposing the step to be elastic:

$$\boldsymbol{\varepsilon}_{n+1}^{p,TR} = \boldsymbol{\varepsilon}_n^p, \tag{5.5.11}$$

$$\boldsymbol{\sigma}_{n+1}^{dev,TR} = 2G[\boldsymbol{\varepsilon}_{n+1}^{dev} - \boldsymbol{\varepsilon}_n^p], \tag{5.5.12}$$

$$\boldsymbol{\sigma}_{b_{n+1}}^{TR} = \boldsymbol{\sigma}_{bn}, \tag{5.5.13}$$

$$\mathbf{s}_{n+1}^{TR} = \boldsymbol{\sigma}_{n+1}^{dev,TR} - q_{n+1}^{TR}, \tag{5.5.14}$$

$$\gamma_{n+1}^{TR} = \gamma_n. \tag{5.5.15}$$

Next step is to check is $\mathbf{s}_{n+1}^{TR}$ admissible:

$$\left\|\mathbf{s}_{n+1}^{TR}\right\| \leq \sigma_{y,0} + \mathbf{H}_{iso}\gamma_{n+1}^{TR}, \tag{5.5.16}$$

where the variable values at the time step t_{n+1} are taken as the trial ones just calculated.

When the previous relation is satisfied one say that step is elastic and the calculation is over. Although, if $\mathbf{s}_{n+1}^{TR}$ violates the yield limit, a plastic correction is introduced:

$$\boldsymbol{\varepsilon}_{n+1}^{p} = \boldsymbol{\varepsilon}_{n+1}^{p,TR} + \lambda\boldsymbol{n}, \tag{5.5.17}$$

$$\boldsymbol{\sigma}_{n+1}^{dev} = \boldsymbol{\sigma}_{n+1}^{dev,TR} - 2G\lambda\boldsymbol{n}, \tag{5.5.18}$$

$$\boldsymbol{\sigma}_{b_{n+1}} = \boldsymbol{\sigma}_{b_{n+1}}^{TR} + H_{kin}\lambda\boldsymbol{n}, \tag{5.5.19}$$

$$\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{TR} - [2G + H_{kin}]\lambda\boldsymbol{n}, \tag{5.5.20}$$

$$\gamma_{n+1} = \gamma_{n+1}^{TR} + \lambda, \tag{5.5.21}$$

where the scalar λ represents the increment of the plastic consistency parameter i.e. $\int_{t_n}^{t_{n+1}} \dot{\gamma} dt$.

In the end one has a system of algebraic equations. To solve this system it is enforced condition $\phi(\mathbf{s}) = 0$. Using property that $\mathbf{s}_{n+1}^{TR}$ and $\mathbf{s}_{n+1}$ are parallel, one can obtain:

$$\lambda = \frac{\left\|\mathbf{s}_{n+1}^{TR}\right\| - (\sigma_{y,0} + \mathbf{H}_{iso}\gamma_{n+1}^{TR})}{2G + \mathbf{H}_{iso} + \mathbf{H}_{kin}}. \tag{5.5.22}$$

Now it is simple to get all variables if it is known that:

$$\boldsymbol{n}^{TR} = \frac{\mathbf{s}_{n+1}^{TR}}{\left\|\mathbf{s}_{n+1}^{TR}\right\|} = \boldsymbol{n}. \tag{5.5.23}$$

5.6 Plastic Models

Current approach to elastoplastic modelling is based on finite numbers of existing yield functions, flow rules and evolution laws. We will not describe it, but only give a short overview. The yield functions are usually based on some existing shapes like von Mises, Tresca, Drucker-Prager, Mohr-Coulomb, Cam Clay [143, 164] and other models. All these models assume linear elastic perfectly plastic brittle weakening behaviour. In other words, the stress-strain response assumes that the material behaves in a linear elastic manner, according to the elastic parameters specified in the model, prior to yielding and in a perfectly plastic behaviour after yielding.

Von Mises criteria is typically used for metal plasticity, while Mohr-Coulomb and Drucker-Prager are typically used for soils and other frictional materials. The plasticity models can freely be combined; the combination of the plasticity surfaces defines the total plasticity surface.

We will give just criteria of these models, all other information can be found in literature [6, 23, 96, 108, 143, 164, 165, 174, 191].

5.6.1 Yield Criteria for Perfect Plasticty- von Mises Criteria

The yield criteria is obtained by extension of the perfect plasticity criteria [14, 42, 164]. In other words perfect plasticity criteria is very important for elastoplastic problems. For perfect plasticity there are no internal variables $\boldsymbol{\vartheta}$, as well as internal force $\boldsymbol{\chi}$, so $\boldsymbol{\Sigma}$ becomes $\boldsymbol{\sigma}$ and yield function is $\phi(\boldsymbol{\sigma})$.

The yield function in the case of isotropic materials is dependent only on the scalar invariants of $\boldsymbol{\sigma}$. Instead of representing yield function as a function of these invariants one can express it as a function of invariants of a deviatoric stress $\boldsymbol{\sigma}^{dev}$:

$$\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \frac{1}{3}(tr\boldsymbol{\sigma})I, \tag{5.6.1}$$

so that the function is given as

$$\phi = \phi(I_2(\boldsymbol{\sigma}^{dev}), I_3(^{dev})) \tag{5.6.2}$$

without first invariant of deviatoric stress which vanish.

The simplest criteria which can be used is von Mises yield criteria [40]. Total elastic energy stored in a unit volume element of the material is given as a sum of deviatoric energy (energy of elastic distortion) and hydrostatic energy (energy of volume change). Metallic materials are found to be insensitive to plastic yielding under hydrostatic stresses, so the criterion is based on the assumption that metal yield when the energy of elastic distortion attains a critical value independently of the particular combination of the stresses (given by uniaxial tensile test) with elastic limit stress $\boldsymbol{\sigma}_s$. Equating this two energies one can get equivalent deviatoric stress

$$\bar{\sigma}^2 = \frac{3}{2}\boldsymbol{\sigma}^{dev^T}\boldsymbol{\sigma}^{dev} \tag{5.6.3}$$

or yield criteria:

$$\bar{\sigma} - \sigma_s \leq 0. \tag{5.6.4}$$

In the component form this criterion is given by

$$\bar{\sigma} = \sqrt{\frac{3}{2}((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2))} \tag{5.6.5}$$

5.6.2 The Tresca Model

In the case of von Mises yield criteria, described yield surface is smooth, although this is not often the case. Therefore it is developed the Tresca yield criterion, describing non-smooth yield surface. It is piecewise smooth, in the sense that it is made up of smooth manifolds whose intersections form corners or vertices. This criterion is based on the assumption that the elastic threshold is reached when the maximum shear stress reaches a particular value. In the terms of principal values σ_i, σ_j and assuming incompressibility and isotropy, the Tresca yield surface is defined by

$$\phi(\sigma) = \max|\sigma_i - \sigma_j| - \sigma_0 = 0, \tag{5.6.6}$$

where σ_0 is the yield stress, obtained from one dimensional tension experiments. In the coordinate system defined by principal stress axes this criterion has a shape of hexagonal prism with its axes equally inclined to the three principal axes.

5.6.3 General von Mises Criteria

In the case of anisotropy von Mises yield criteria becomes general and it cannot be expressed as a function of a just scalar invariants. Instead, in the story comes $\mathbf{D}$ a fourth order tensor possessing symmetries:

$$D_{ijkl} = D_{klij} = D_{jikl}. \tag{5.6.7}$$

General von Mises criteria then has a form

$$\phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{D}\boldsymbol{\sigma} - k^2 = 0. \tag{5.6.8}$$

5.6.4 Hill's Model

Hill's model is a special case of von Mises general model, assuming existence of three mutually perpendicular planes of symmetry. The basis is chosen to be such that three basis vectors are normal to the three planes of symmetry. In that case $\mathbf{D}$ has only nine independent components and yield criteria has the form:

$$\boldsymbol{\sigma} : \mathbf{D}\boldsymbol{\sigma} = D(\sigma_{22} - \sigma_{33})^2 + B(\sigma_{11} - \sigma_{33})^2 + C(\sigma_{11} - \sigma_{22})^2 + D\sigma_{23}^2 + E\sigma_{13}^2 + F\sigma_{12}^2, \tag{5.6.9}$$

where $D_{1111} = B + C, D_{1122} = -C, D_{1133} = -B$, and so on.

5.6.5 Dracker-Prager and Cam-Clay Model

Dracker-Prager yield function with linear mixed hardening is given:

$$\phi = 3\alpha p^{'} + \sqrt{J_2^{'}} - c - h_i\gamma, \tag{5.6.10}$$

where parameters c and α are related to friction angle and cohesion of a geomaterial. The parameter $h_i\gamma$ accounts the isotropic hardening and γ is integral of plastic multiplier. $J_2^{'}$ is the second invariant of the stress with kinematic hardening.

For the Modified Cam Clay model [143] all stresses are effective (geotechnical) stresses, i.e. compression is positive. Yield function is given as:

$$\phi = q^2 - M^2[p(p_0 - p)] = 0, \tag{5.6.11}$$

where p is a mean stress and q is given:

$$q = \{\frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2) + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)]\}^{\frac{1}{2}}, \tag{5.6.12}$$

where p_0 is history (hidden) variable which corresponds to the preconsolidation mean pressure, while M is a soil constant.

5.7 Conclusion

Since the objective of this review is to present the theory of stochastic elastoplasticity, an attempt is made to formulate the basic governing equations in a concise form. First we give the general theory of plasticity which can be used in all the cases of plasticity, as well as classical constitutive equations. In more mathematical way has been given the primal and dual problem of plasticity and their connection with the classical formulation.

In this chapter we give formulation and the numerical implementation of the plastic models. First, it is described the model of perfect plasticity, in the meaning of no hardening, then both types of hardening, kinematic and isotropic. The main difference among them are that in the case of kinematic hardening one has the moving of the centre of the yield surface, while in the case of the isotropic hardening the centre doesn't move but only the radius of the yield surface changes.

Well established and performing algorithm for dealing with the problems of plasticity is the radial return map algorithm, which is also described. In general, the method consists of the time integration of the differential algebraic system leading to an algebraic system, and of the formulation of the solution algorithm for the obtained algebraic system.

In practice exist a lot of models of yield criteria used to describe plasticity. Here are given in very short form perfectly plastic material models and some of those describing soils.

Chapter 6

Stochastic Methods

In the process of studying stochastic differential equations (SPDE) the numerical simulations play very important role [94]. There are few numerical methods dealing with the process of simulation of SPDE. They are classified as methods of the moment equations, the probability density function, the Monte Carlo methods [75] and etc. The review of development of computational procedures as utilised in stochastic mechanic is given by Schüeller [155].

6.1 Stochastic Problems

Let the system be stationary without time dependence and the time derivatives in the form:

$$Au = f \tag{6.1.1}$$

in bounded domain $\mathcal{G}$ in $\mathbb{R}^k$, where A is an operator, f is given in $L^2(\mathcal{G})$ and initial conditions are given in the form:

$$u = 0 \tag{6.1.2}$$

on Γ (boundary of domain $\mathcal{G}$). Let operator A be defined as:

$$Av = -\mathrm{div}(\kappa \nabla v), \tag{6.1.3}$$

where κ is a real stochastic function [7].

When A is stochastic operator our system becomes the stochastic system, described with stochastic partial differential equations [145]. In deterministic case system is well-posed in the sense of Hadamard. That means that solution depends continuously on the right hand side in the norm of L^2 and on operator κ, but in the norm L^∞. For this to hold, it is necessary that the operator κ is continuous and continuously invertible. To achieve this in stochastic case κ must satisfy:

$$0 < \kappa(x, \omega) < +\infty. \tag{6.1.4}$$

Extending all other conditions into the stochastic domain, it is possible to show that problem is well-posed.

Stochastic partial differential equations are classified as partial differential equations with stochastic right hand side - additive noise or with stochastic operator - multiplicative noise [32]. The first case (the stochastic right hand side) is very similar to the linear time-invariant dynamical systems with random inputs, where one uses Fourier transform and observes the transfer function of the system. Instead of Fourier transform Matthies used the Karhunen-Loève expansion [120, 121].

Variational formulation is not only characteristic of deterministic system, but also of the system with uncertainty. It looks for the solution u in a space $\mathcal{V} \otimes \mathcal{S}$ of functions with spatial dependence (the variables from $\mathcal{V}$) and stochastic dependence (the variables from $\mathcal{S}$) such that for all $v \in \mathcal{V} \otimes \mathcal{S}$ is fullfilled:

$$b(u, \omega) = \langle\langle f, v \rangle\rangle, \tag{6.1.5}$$

where

$$b(u, \omega) = \int_\Omega \int_\mathcal{G} (\nabla v(x, \omega))^T \kappa(x) (\nabla u(x, \omega)) dx dP(\omega) \tag{6.1.6}$$

and

$$\langle\langle f, v \rangle\rangle = \int_\Omega \int_\mathcal{G} f(x, \omega) v(x, \omega) dx dP(\omega). \tag{6.1.7}$$

In both cases, the stochastic right hand side and the stochastic operator (or both together), the spatial part of the SPDE is approximated with a Galerkin method [32].

For the finite dimension subspace $\mathcal{V}_N \in \mathcal{V}$ and a basis $s_1(x), .., s_N(x)$ in that subspace, the solution can be approximated as [29, 43]:

$$u(x, \omega) = \sum_{k=1}^{N} s_k(x) u_k(\omega) = \boldsymbol{s}(x) \boldsymbol{u}(\omega), \tag{6.1.8}$$

where $u_k(\omega)$ are the random variables in $\mathcal{S}$. The variational form in that case is :

$$\int_\Omega \varphi(\omega) \boldsymbol{K} \boldsymbol{u}(\omega) dP(\omega) = \boldsymbol{K} \int_\Omega \varphi(\omega) \boldsymbol{u}(\omega) dP(\omega) = \int_\Omega \varphi(\omega) \boldsymbol{f}(\omega) dP(\omega), \tag{6.1.9}$$

where:

$$\boldsymbol{K} = (K_{ij}) = (\int_R (\nabla s_i(x))^T \kappa(x, \omega) (\nabla s_j(x)) dx) \tag{6.1.10}$$

represents the stiffness matrix. The right hand side then is given as

$$\boldsymbol{f}(\omega) = [f_1(\omega), .., f_N(\omega)]^T \tag{6.1.11}$$

with components:

$$f_j(\omega) = \int_R s_j(x) f(x, \omega) dx. \tag{6.1.12}$$

The variational form of equilibrium equation now can be written as:

$$\mathbf{K}(\omega)\boldsymbol{u}(\omega) = \boldsymbol{f}(\omega) \tag{6.1.13}$$

and it has similar form like in the deterministic case. Also, the stiffness matrix in the case of stochastic right hand side is deterministic, while for multiplicative noise and for combination of additive and multiplicative noise this is not the case.

For solving stochastic partial differential equations several different numerical methods are developed. They are reviewed in next chapters in short notes.

6.2 Monte Carlo Method

The name of this method has no significance. First was used by von Neumann during the World War II as a code word for secret work on nuclear weapons at Los Alamos Laboratory in New Mexico.

The major advantage of Monte Carlo simulation is that accurate solutions can be obtained for any problem whose deterministic solution is known. The only disadvantage of Monte Carlo simulation is that it is generally time-consuming. It is based on generation of sample functions of the stochastic processes and fields involved in the problem. The generated sample functions must accurately describe the probabilistic characteristics of the corresponding stochastic processes and fields that may be either stationary or non-stationary, homogeneous or non-homogeneous, one-dimensional or multidimensional, univariate or multivariate, Gaussian or non- Gaussian.

Monte Carlo method consist of several essential steps [66]:

- defining the problem in terms of all the random variables,
- quantifying the probabilistic characteristics of all the random variables in terms of their probability density functions and the corresponding parameters,
- generating values of these random variables,
- evaluating the problem deterministically for each set of realizations of all the random variables ,
- extracting probabilistic information from N such realizations,
- determining the accuracy and efficiency of the simulation.

As one can notice, numerical methods that are known as Monte Carlo methods can be loosely described as statistical simulation methods, where statistical simulation is defined in quite general terms to be any method that utilises sequences of random numbers to perform the simulation. Statistical simulation methods may be contrasted to conventional numerical discretisation methods, which typically are applied to ordinary or partial differential equations that describe some underlying physical or mathematical system. In

many applications of Monte Carlo, the physical process is simulated directly, and there is no need to even write down the differential equations that describe the behaviour of the system. The only requirement is that the physical (or mathematical) system be described by probability density functions (pdf's).

Monte Carlo method is robust since the accuracy only depends on the crudest measure of the complexity of the problem. But the prize of its robustness is a high computational effort i.e. extremely slowness [20]. To accelerate the convergence rate one can use variance reduction methods using random or pseudo random sequences.

Monte Carlo method is wide used for calculating the ultimate goal of process of solving stochastic system i.e. response statistics (functionals of the solution):

$$\Psi_u(x) = \mathbf{E}(\Psi(x, \omega, u(x, \omega)) = \int_\Omega \Psi(x, \omega, u(x, \omega)) dP(\omega). \tag{6.2.1}$$

Given integral can be approximated by a weighted sum of samples of the integrand [120], wherefore it is very simply and easy to use the Monte Carlo method. Its convergence is independent of the dimension of the integral and convergence rate is about $O(N^{-1/2})$. Due to the numerical treatment of the integral, domain Ω must be replaced with the finite dimensional Θ_m:

$$\Psi_{iu}(x) \approx \sum_{z=1}^{Z} \omega_z \Psi(x, \theta_z(\omega), s(x), u(\theta_z(\omega))) \tag{6.2.2}$$

where s is the spatial basis [120], θ_z are evaluation points and ω_z are weights, which are equal to 1/Z. Process of selecting the evolution points is according to the integration rule. For each evaluation point one has to solve the deterministic problem with fixed realization, compute the previous integrand and in the end perform the summation.

The convergence of Monte Carlo method is according to the strong law of the large numbers given by probability one:

$$\lim_{Z \to \infty} \Psi_{iu} - \Psi_u, \tag{6.2.3}$$

while integration error is described by

$$\varepsilon_N[f] = \Psi_{iu} - \Psi_u. \tag{6.2.4}$$

For large number of realisations N size of the error and statistical properties are described by the central limit theorem [20]:

$$\varepsilon_N[f] \approx \sigma N^{-1/2} \nu \tag{6.2.5}$$

where ν is a standard normal variable ($\mathcal{N}(0, 1)$) and σ standard deviation of the Ψ. This error is probabilistic and it is of a certain size with some probability.

As we said before, the main advantage of Monte Carlo method is that it does not depend of the integration domain like the other methods are affected. But the main disadvantage is that it's very slow. The computational time grows rapidly as the desired accuracy is

tightened. To reduce it one can use the variance reduction in which the integrand is transformed so that σ is reduced or to replace the random variables with the alternative sequence improving exponent 1/2. Some of often used methods are : antithetic variables method (add's integration points at $-\omega_i$), control variates, matching moment methods and other many methods which can be found in the literature [20].

6.3 Quasi Monte Carlo Method

To reduce computational effort of Monte Carlo method it is exploited quasi-Monte Carlo method [20]. Quasi-Monte Carlo methods use the quasi random sequences and evaluate integral using the correlation points. The sequence is called quasi-random if it is:

$$D_Z \leq c(\log N)^k N^{-1} \tag{6.3.1}$$

where c and k are constants that are independent of N but may depend of the dimension m i.e. dimension of the sequence, while correlated points are generated from so-called low discrepancy series.

The error of quasi Monte Carlo method is given by Koksma-Hlawka inequality [20], describing upper bound of the error:

$$\varepsilon \leq V(\psi) D_Z, \tag{6.3.2}$$

where $V(\Psi)$ is the total variation of the integrand and D_Z the factor which depend of the sequence i.e. discrepancy of the series. The Koksma-Hlawka inequality and discrepancy bounds for the quasi-random sequence together imply faster convergence then the standard Monte Carlo method. The resulting convergence rate is $O((\log Z)^k Z^{-1})$. In the case of high dimensions the logarithm term in error dominates, while for many types of integrands it's only $O(Z^{-1})$ [157].

Comparing to the standard Monte Carlo method error, one can conclude that both methods have the same type of error, given with product of two coefficients (first depending of the sequence and second which depending on the integrand Ψ). Error inequality is the worst case inequality for quasi Monte Carlo method, while in the case of standard Monte Carlo method it is a probabilistic bound. In the Monte Carlo method each point is the estimate of the integral while in quasi method initial points sample the integrand on the coarse scale and later on the finer scale. Also total variation overestimates the error while the discrepancy is good estimator for the actual error.

6.4 Perturbation Method

This method is probably the most popular method for solving stochastic partial differential equations. Description of this method is given by Shinozuka, Hisada, Kleib and many other authors [34, 71, 93, 162].

Perturbation method is based on approximation of the stochastic partial differential equation in a finite number of random variables. It uses Taylor expansion of the solution in the basic independent random variables up to first or second order. The perturbation approach represents the stochastic response like a small variation about the mean. Assuming that the stochastic parameters vary a little bit around their mean, perturbation approach will give the variation of the response about the mean.

Disadvantage of this method is that expansion is only of order two or less and only first and second order statistics are computed. There is possibility to calculate moments higher then second but then its hard to work with complicating expressions. Next disadvantage of this method is that only permit small deviations of the mean value.

Perturbation methods used by several authors [18, 38, 171] are difficult to implement for orders higher than two, and are theoretically restricted to coefficient of variation less than one [190]. In terms of coefficient of variation, cv (ratio between the standard deviation and the average value), it is generally accepted that perturbation series converges for coefficients of variation less than one [13, 113]. Note that even when this condition of convergence is fulfilled, the accuracy of a first order approach solution limits the variation of the coefficients of variation in the range of 1. The perturbation method requires derivatives of the system matrix and of the right hand side with respect to the random variables, which is difficult to apply or it needs some automated derivation.

The system matrix is represented in the series:

$$\boldsymbol{K} = \boldsymbol{K}^0 + \sum_{i=1}^{N} \boldsymbol{K}_i^I \xi_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{i=1}^{N} \boldsymbol{K}_{ij}^{II} \xi_i \xi_j + ... \tag{6.4.1}$$

where α are random variables and :

$$\boldsymbol{K}_i^I = \frac{\partial \boldsymbol{K}}{\partial \xi_i}|_{\xi=0} \tag{6.4.2}$$

$$\boldsymbol{K}_{ij}^{II} = \frac{\partial^2 \boldsymbol{K}}{\partial \xi_i \partial \xi_j}|_{\xi=0}. \tag{6.4.3}$$

If ξ are not Gaussian random variables one must evaluate higher-order moments for the second-order approximation in accordance with their joint probability distribution. But very often the calculation of these higher order moments is impossible.

6.5 Neumann Series Method

It should point out that the Neumann decomposition is not a perturbation method; no small parameter is involved in the algorithm of the method. Utilisation of formal series is not restricted by the condition of convergence ($cv << 1$) of the perturbation series [24, 190]. However, it is not clear, whether the convergence of the Neumann series is restricted to small variations of the parameters of the system.

Finding solution of equilibrium equation is based on finding inverse of the stiffness matrix, which induce a long computational time. Also, the inverse matrix is no longer banded although the stiffness matrix is. The Neumann expansion technique has been adopted to avoid the repeated inversion of the random stiffness matrix.

The Neumann expansion method is based on the decomposition of stiffness matrix on two parts-first which represent the part in which spatially variable parameters are replaced by their mean values $\boldsymbol{K}_0$ and deviatoric part given by [162]:

$$\triangle \boldsymbol{K} = \boldsymbol{K} - \boldsymbol{K}_0. \tag{6.5.1}$$

Then Neumann expansion is given by :

$$\boldsymbol{K}^{-1} = (\boldsymbol{K}_0 + \triangle \boldsymbol{K})^{-1} = (1 - P + P^2 - P^3 + ...)\boldsymbol{K}_0^{-1} \tag{6.5.2}$$

with :

$$P = \boldsymbol{K}_0^{-1} \triangle \boldsymbol{K}. \tag{6.5.3}$$

6.6 Stochastic Galerkin Method

In Chapter 4. we already explained the spectral stochastic method of the random fields. Here we will give short explanation of this method, used for solving stochastic problems. The spectral stochastic finite element method (SSFEM) was proposed first by Ghanem and Spanos [53, 54, 55, 56, 57, 58, 59]. The stochastic system is governed by a set of stochastic partial differential equations in a sense that response of the system is given as $u(x, \omega)$ where ω represents a basic outcome in the space of all possible outcomes. SSFEM discretise random dimension using series expansions:

- the input is discretised using the Karhunen-Loève expansion (Chapter 4.)
- displacement is represented in appropriate basis of the space of the random variables (polynomial chaos).

6.6.1 Polynomial Chaos Expansion

Ghanem and his co-workers [59, 57, 53, 55, 58, 56, 54] proposed expansion of the system matrix using polynomial chaos expansion:

$$\boldsymbol{K}(\theta) = \sum_l \boldsymbol{K}^{(l)} H_l(\theta), \tag{6.6.1}$$

where coefficients of the expansion are given

$$\boldsymbol{K}^{(l)} = \|H_l(\theta)\|^{-2} \mathbf{E}(\boldsymbol{K}(\theta) H_l(\theta)). \tag{6.6.2}$$

Every matrix term in this expansion $K^{(l)}$ can be computed using the deterministic code with $\kappa^{(l)}(x) = \|H_l(\theta)\|^{-2}\mathbf{E}(\kappa(x,\theta)H_l(\theta))$ as a material parameter, usually calculated at the integration points.

As Kesse pointed out [88, 89], projection of a random field κ, given as the transformation of the Gaussian random field $\kappa(x,\theta) = \phi(x,\gamma(x,\theta))$, can be computed using Karhunen-Loève expansion of the Gaussian random field $\gamma(x,\theta) = \sum_{i=1}^{\infty}\sqrt{\gamma_i}\gamma_i(x)\theta_i$, where γ_i are the eigenvalues and γ_i the eigenfunctions of the eigenvalue problem. Using the definition of the coefficients of polynomial chaos expansion in the case when functions are smooth one can write:

$$\kappa^{(l)} = \|H_l(\theta)\|^{-2}\mathbf{E}(\kappa(x,\theta)H_l(\theta)) = \frac{1}{l!}\mathbf{E}(\kappa(x,\theta)H_l(\theta)) \tag{6.6.3}$$

i.e.

$$\kappa^{(l)} = \frac{1}{l!}\mathbf{E}(\frac{\partial^{|l|}}{\partial\gamma^{|l|}}\phi(x,\gamma)|_{\gamma=\gamma(x,\omega)}\prod_{i=1}^{\infty}(\sqrt{\gamma_i}\gamma_i(x))^{l_i}. \tag{6.6.4}$$

In practice this expression is very easy to calculate for different types of marginal distributions. Also, to enable numerical calculation one needs truncation of the polynomial chaos expansion after certain number of terms m:

$$\boldsymbol{K}(\theta) = \sum_{l=0}^{m}\boldsymbol{K}^{(l)}H_l(\theta). \tag{6.6.5}$$

The number of terms must be enough such that matrix $\boldsymbol{K}$ has exact representation. Expanding soluton $\boldsymbol{u}$ and right hand side in polynomial chaos expansion and puting back into final equilibrium equation, one has

$$\sum_{l=0}^{m}\boldsymbol{K}^l H_l(\theta)\sum_{\alpha=0}^{m}\boldsymbol{u}^{(\alpha)}H_\alpha(\theta) = \sum_{\beta=0}^{m}\boldsymbol{f}^{(\beta)}H_\beta(\theta). \tag{6.6.6}$$

The system matrix $\boldsymbol{K}$ through Galerkin projection can be represented using sub-matrices $\boldsymbol{K}_{\alpha,\beta}$:

$$\boldsymbol{K}_{\alpha,\beta} = \sum_{l}\mathbf{E}(H_\alpha(\theta)H_\beta(\theta)H_l(\theta))\boldsymbol{K}^l. \tag{6.6.7}$$

It can be shown that this sum over l is finite [88]. Denoting with

$$\triangle^{(l)}_{\alpha,\beta} = \mathbf{E}(H_\alpha(\theta)H_\beta(\theta)H_l(\theta)), \quad \alpha,\beta\in\mathcal{I}, \tag{6.6.8}$$

matrix $\boldsymbol{K}$ may be written as a sum of Kronecker products in a way

$$\mathbf{K} = \sum_{l}\triangle^{(l)}\otimes\boldsymbol{K}^{(l)}. \tag{6.6.9}$$

Finally, system of equations will have the structure of:

$$\mathbf{Ku} = \sum_{l}[\triangle^{(l)}\otimes\boldsymbol{K}^{(l)}]\mathbf{u} = \mathbf{f} \tag{6.6.10}$$

i.e.

$$\sum_l \begin{pmatrix} \triangle^{(l)}_{1,1}K^{(l)} & \dots & \triangle^{(l)}_{1,|I|}K^{(l)} \\ \dots & \dots & \dots \\ \triangle^{(l)}_{|I|,1}K^{(l)} & \dots & \triangle^{(l)}_{|I|,|I|}K^{(l)} \end{pmatrix} \begin{pmatrix} u^{(1)} \\ \dots \\ u^{(|I|)} \end{pmatrix} = \begin{pmatrix} f^{(1)} \\ \dots \\ f^{(|I|)} \end{pmatrix}. \tag{6.6.11}$$

The stiffness matrix is a full block matrix. Therefore, it must be efficiently used in a way that its storing may not be of high cost. Hence, Kronecker product representation reduces memory requirements [44].

6.6.2 Karhunen-Loève Expansion

Like it is shown, the Karhunen-Loève expansion of the stiffness matrix, may be represent as:

$$\boldsymbol{K}(\omega) = \overline{\boldsymbol{K}} + \sum_{j=1}^{\infty} \varsigma_j \xi_j(\omega) \boldsymbol{K}_j, \tag{6.6.12}$$

where $\boldsymbol{K}_j$ is computed by using the KL-eigenfunction $\kappa_j(x)$ and ς_j, ξ_j, which are the singular values and the eigenfunctions of the KLE of $\kappa(x,\omega)$. Using the polynomial chaos expansion as before and imposing Galerkin conditions one obtains:

$$\forall \alpha \in J_{M,p} : \sum_{\beta \in J_{M,p}} \mathbf{E}(H_\alpha(\cdot)\boldsymbol{K}(\cdot)H_\beta(\cdot))\boldsymbol{u}^{(\beta)} = \mathbf{E}(H_\alpha(\cdot)f) = \boldsymbol{f}^{(\alpha)}. \tag{6.6.13}$$

Expanding $\boldsymbol{K}(\cdot)$ in the KL :

$$\sum_{\beta \in J_{M,p}} \mathbf{E}(H_\alpha(\cdot)\boldsymbol{K} + \sum_{j=1}^{\infty} \varsigma_j \xi_j(\omega)\boldsymbol{K}_j(\cdot)H_\beta(\cdot))\boldsymbol{u}^{(\beta)} = \boldsymbol{f}^{(\alpha)} \tag{6.6.14}$$

and again using the PCE of $\xi(\omega) = \sum_\gamma c_j^{(\gamma)} H_\gamma(\omega)$ one obtains solvable equation:

$$\mathbf{Ku} = [\sum_{j=0}^{\infty} \triangle^{(j)} \otimes \boldsymbol{K}_j]\mathbf{u} = \mathbf{f} \tag{6.6.15}$$

where $\triangle^{(j)} = \sum_{\gamma \in J_{M,2p}} \varsigma_j c_j^{(\gamma)} \triangle^{(\gamma)}$. In [122] is given theorem:

Theorem 6.6.1. *The series in Eq. (6.6.15) converges uniformly. Hence a finite number of terms suffices to keep the discrete operators* $\mathbf{K}$ *uniformly in the discretisation of* $\mathcal{V} \otimes \mathcal{S}$ *positive definite and therefore their inverses uniformly bounded assuring the stability of the approximation process.*

When we have $\boldsymbol{u}^\alpha$ all the statistics can be computed approximately and especially covariance:

$$\mathbf{C}_y(x,y) \approx \boldsymbol{s}(x)\mathbf{E}(\widetilde{\boldsymbol{u}}(\omega) \otimes \widetilde{\boldsymbol{u}}(\omega))\boldsymbol{s}^T(y) = \boldsymbol{s}(x)\mathbf{C}_u\boldsymbol{s}^T(y), \tag{6.6.16}$$

where :

$$\mathbf{C}_u = \mathbf{E}(\widetilde{\boldsymbol{u}}(\omega) \otimes \widetilde{\boldsymbol{u}}(\omega)) = \sum_{\alpha \in J_{M,p}} \boldsymbol{u}^{(\alpha)} \otimes \boldsymbol{u}^{(\alpha)}. \tag{6.6.17}$$

The polynomial expansion can be used but only when it is used the Galerkin method, otherwise not directly or without other precautions.

6.6.3 Numerical Strategies

As it is shown in previous chapters, using stochastic finite element approach one get the system of equations:

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{6.6.18}$$

where $\mathbf{K}$ is the stiffness matrix, $\boldsymbol{u}$ is a displacement and $\boldsymbol{f}$ are external forces. On the first look this system of equations is the same as the system of equations for deterministic problem. But it is not. The problem is that this system of equations is much more larger then a spatial equivalent system of deterministic equations. The inclusion of additional interpolation functions in the random dimensions coupled with the numerous random variables, required to represent the involved random fields, dramatically increases the number of equations. If it is used direct approach to solve this system, not only that full assembled system matrix must be stored but also a large part of the memory must be dedicated to additional fill-ins generated during the solution process. Therefore, direct solution process is usefulness.

Iteration algorithms do not require storage of the full assembled system matrix. They are based on matrix-vector product evaluations [28]. Therefore, less computer storage is needed since these products are calculated using smaller commponent matrices. Consequently iterative algorithm has a distinct advantage of solving large system of equations. In other words this approach is based on finding approximate solution by minimizing the residue:

$$\mathbf{r}^{iter} = \mathbf{f} - \mathbf{K}\mathbf{u}^{iter}. \tag{6.6.19}$$

Here we do not need the stiffness matrix $\mathbf{K}$, but only calculation of the matrix-vector product in each iteration. Chuang and Gutierrez [27] gave the short review of this type of methods. There are a few different iterative approaches described like conjugate gradient method, spectral conjugate gradient method, element by element gradient method etc.

The conjugate gradient method calculate the matrix-vector product $\mathbf{K}\mathbf{u}^{iter-1}$ and use it in current iteration. It iteratively minimizes the residue vector by appdating the approximate solution $\mathbf{u}^{iter}$ using a search vector $\mathbf{p}^{iter}$. The efficiency depends of the time needed for computing the matrix-vector product and convergence, which depends of the condition number, defined as the ratio between the largest and smallest eigenvalues of the $\mathbf{K}$. To improve this system of equations one can use transformation into a system with lower condition number.

In the spectral conjugate method the internal force vector is calculated as the matrix-vector product:

$$\mathbf{t}_k = \mathbf{K}\mathbf{u}^{iter} = \sum_{i=1}^{L}\sum_{j=1}^{P} c_{ijk}\boldsymbol{K_i}\boldsymbol{u}_k^{iter}. \tag{6.6.20}$$

Storing just non-zero elements of c_{ijk} and $\boldsymbol{K}_i$ one can decrease the computational time. The usage of the spectral component matrices in the computation of the residual vector involves the storage of the matrices $\boldsymbol{K}_i$ and the preconditioning matrix Δ. The storage capacity for these matrices grows linearly with the number of terms retained in the KL expansion and this limits the number of terms that can be considered in the spectral stochastic finite element analysis.

The element by element conjugate method as the name says computes the internal force vector using element contributions:

$$\mathbf{t}_k = \mathbf{K}\mathbf{u}^{iter} = \sum_{e=1}^{E} \boldsymbol{K}^e(\boldsymbol{u}^e)^{iter}. \tag{6.6.21}$$

Thsi kind of evaluation requires only the temporary usage of the single element matrix, however the element matrix for each element must be re-assembled before matrix-vector product evaluation. So, the fully assembled system matrix is not available and the system cannot be preconditioned. This leads to slow convergence and long execution time for assembling the system matrix.

The block Gauss-Jacob method is a method with numerical procedure given by algorithm:

$$\hat{\mathbf{K}}\delta\mathbf{u} = \mathbf{f} - \mathbf{t}^{iter} \tag{6.6.22}$$

$$\mathbf{u}^{iter+1} = \mathbf{u}^{iter} + \delta\mathbf{u}^{iter} \tag{6.6.23}$$

where $\hat{\mathbf{K}}$ is the diagonal block form of the system matrix and $\mathbf{t}$ is internal force vector. This new system of equations can be solved using conjugate gradient method. However, this system is smaller then that one in beginning, so it can be solved in fewer iterations and with less computation time.

6.7 Hybrid SSFEM

The hybrid SSFEM represents coupling between Monte Carlo simulation and stochastic finite element method. It is proposed by Ghanem [60]. The idea behind this method is to use truncated polynomial chaos expansion of the response, representing residual in terms of a set of delta functions $\Phi(\theta) = \delta(\theta - \theta_j)$:

$$U(\theta) = \sum_{j=0}^{m} u^{(j)} H_j(\theta) + \sum_{i=0}^{(q-1)} u_*^{(i)} \Phi_i(\theta). \tag{6.7.1}$$

Substituting in the equilibrium equation obtained residual is made orthogonal to the both H_j and Φ_i. Obtained system of equation then is solved iteratively at lower cost then direct approach.

6.8 Stochastic Response Surface Method

The stochastic response surface method (SRMS) is similar to deterministic one, where instead of deterministic inputs one has a vector of random variables. The main idea is to evaluate desired response in some chosen points of stochastic space and then to fit these response points with some analytic expression (which is often a low order polynomial). In other words the aim of the SRSM is to estimate the statistical moments of the output variable by using a reasonable low number of function calls. This is achieved by expanding both, the input and output variable, in terms of standard normal variables. The series expansion for the output variable contains unknown coefficients which can be determined through a reasonably small number of function calls.

The implementation consist in representing the input variables in terms of a set of standard normal variables, the output as a specific expansion and determining parameters of that expansion. Let κ be a vector of random variables., then $\mathrm{g}(\boldsymbol{\kappa})$ will be function of output, which is usually not know explicitly, so that in practice are often used quadratic functions instead:

$$\mathrm{g}(\boldsymbol{\kappa}) \approx \hat{\mathrm{g}}(\boldsymbol{\kappa}) = a_0 + \sum_{i=1}^{N} a_i \kappa_i + \sum_{i=1}^{N} a_{ii} \kappa_i^2 + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} a_{ij} \kappa_i \kappa_j, \tag{6.8.1}$$

where the set of coefficients $\{a_0, a_i, a_{ii}, a_{ij}\}$ need to be determined using some of well know fitting techniques, like the least square method.

There are various types of applications of this method. Early implementation is given by Wong [181] using so-called factorial experimental design, where for each random variable one select lower and upper realisations, defining 2^N fitting points as all the possible combinations. Bucher and Bourgund used $2N + 1$ fitting points choosing mean vector of input random variables as the center point of the response surface and then choosing fitting points along the axes around this mean value. Using this response surface new estimate point is computed and then new center is obtained using linear interpolation and so on. All approaches are reviewed in Bruno [168].

For the output can be used also polynomial chaos expansion like it is presented in Anile [5], whose coefficients may be estimated using a variant of the methods of orthogonal disposition i.e. choosing set of points whose components are roots of Hermite polynomial of degree less or equal to the degree of the expansion. But problem can appear when polynomials are of higher degree. In that case one can use regression methods or combination of disposition and regression methods.

Output statistics like average, variance, higher order moments are estimated by generating a large number of standard normal variables, calculating the input values from

the transformation procedure, calculating the output values from the approximation and performing the statistical analysis.

6.9 Method Based on Probability Distribution

Among the methods based on probability distributions one can find method based on Euler-Lagrangian formulation of Fokker-Planck equation [161]. In next chapter we will give mathematical formulation of a method [80, 160, 161]

6.9.1 Fokker-Plank Equation

With the calculus of Ito and Stratonovich one has the mathematical tools to study multiplicative stochastic differential equations, whose solutions are in general Markovian diffusion processes [110]. In this case, instead of studying the stochastic differential equations directly [110], one can also study the corresponding Fokker-Planck equation. There are a few ways to solve Fokker-Planck equation [151]:

- linearization which is the most common way of approximating the nonlinear equations. The idea of this method is to use a mean solution from the nonlinear noise-free model and then to use the linearised model in a covariance propagation;
- second order Taylor series of the function with respect to the mean;
- unscented transformation, using approximation of a first two moments of state distribution
- Interacting Multiple Models (IMM), a special case of Multiple Model Kalman Filter algorithms, well suited to approximating nonlinear dynamic models. The state distribution is represented as a mixture of Gaussian distributions.
- Monte Carlo Methods [141] are common methods in approximating the solutions of partial differential equations such as FPKE. The idea is to discretise the Wiener process (or Brownian motion process), which models the noise in Langevin equation. Then we can simulate sample trajectories from the Langevin equation by using random number generator and Euler integration;
- numeric solving of partial differential equation with the method of Galerkin [65], finite differences (FD) [25] or generalised Edgeworth series.

The Fokker-Planck equation [141] is a powerful method for solving many problems concerning stochastic processes. It usually appears for variables describing a macroscopic but small subsystem in the field of a stochastic force. The FPE was first applied by Fokker and Planck to describe the Brownian motion of particles. It is worthwhile to consider this process more carefully.

First, we speak about the Chapman-Kolmogorov equation which introduce the Markov assumption in terms of the conditional probabilities. If the time satisfies ordering [51, 85] :

$$t_1 \geq \ t_2 \geq \ ... \geq \tau_1 \geq \tau_2 \geq ... \tag{6.9.1}$$

then the conditional probability is determined entirely by the knowledge of the most recent condition [51]:

$$p(x_1, t_1; x_2, t_2; ...|y_1, \tau_1; y_2, \tau_2; ...) = p(x_1, t_1; x_2, t_2; ...|y_1, \tau_1). \tag{6.9.2}$$

Now we can introduce the Chapman-Kolmogorov equation [85] :

$$p(x_1, t_1|x_3, t_3) = \int (x_1, t_1|x_2, t_2)p(x_2, t_2|x_3, t_3)dx_2 p. \tag{6.9.3}$$

The Fokker-Planck equation is a case of Chapman-Kolmogorov equation without the third part of equation which is completely derived in [51]. So it is given by equation:

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x,t)f(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(x,t)f(x,t)] \tag{6.9.4}$$

where A is known as a drift vector and B is called the diffusion coefficient. Mathematically speaking it's a second order partial differential equation of a parabolic type. The diffusion term is result of the stochastic force. We will start from Langevin's equation, which is ordinary differential equation in which a rapidly and irregularly fluctuating random function of time occurs. The simple form of this equation is:

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t), \tag{6.9.5}$$

where a and b are certain known functions and $\xi(t)$ is the rapidly fluctuating random term. This equation must be integrable. That means that next integral exist:

$$u(t) = \int_0^t \xi(t')dt'. \tag{6.9.6}$$

Since the sample functions of the $u(t)$ are continuous, previous equation can be described by the Fokker-Plank equation. Thus this equation is that of the Wiener process [1]and we can write:

$$\int_0^t \xi(t')dt' = u(t) = W(t). \tag{6.9.7}$$

[1]A continuous-time stochastic process $W(t)$ for $t \geq 0$ with $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$ for any $0 \leq s < t$, and increments for nonoverlapping time intervals are independent. Brownian motion (i.e. random walk with random step sizes) is the most common example of a Wiener process.

But previous equation is a paradox because W(t) is not differentiable [51] and mathematically speaking that means that Langevin's equation doesn't exist. However, the corresponding integral equation is:

$$x(t) - x(0) = \int_0^t a[x(s), s]ds + \int_0^t b[x(s), s]\xi(s)ds \tag{6.9.8}$$

and can be interpreted consistently.

This equation can be rewritten as [46]:

$$x(t) = x(0) + \int_0^t a[x(s), s]ds + \int_0^t b[x(s), s]dW(s). \tag{6.9.9}$$

If we have the properties [141]:

$$dW_i dW_j = \delta_{ij} dt, \tag{6.9.10}$$

$$[dW_i]^{N+1} = 0, \tag{6.9.11}$$

$$dW_i dt = 0, \tag{6.9.12}$$

then considering the arbitrary function of $x(t) : f(x(t))$ and expanding in series one obtains the Ito formula [142, 46] :

$$\begin{aligned} df[x(t)] = \{a[x(t), t]f'[x(t)] + \frac{1}{2}b[x(t), t]^2 f''[x(t)]\}dt \\ + b[x(t), t]f'[x(t)]dW(t). \end{aligned} \tag{6.9.13}$$

Using Ito formula connection between Fokker-Planck and stochastic differential equation may be established. Knowing that $x(t)$ has a conditional probability density $p(x, t|x_0, t_0)$ and that $\langle dW_i(t)\rangle = 0$ one has:

$$\begin{aligned} \frac{d}{dt} < f[x(t)] > &= \int dx f(x)\partial_t p(x, t|x_0.t_0) \\ &= \int dx[a(x, t)\partial_x f + \frac{1}{2}b(x, t)^2\partial_x^2 f]p(x, t|x_0, t_0) \end{aligned} \tag{6.9.14}$$

which leads to equation :

$$\partial_t p(x, t|x_0, t_0) = -\partial_x[a(x, t)p(x, t|x_0, t_0)] + \frac{1}{2}\partial_x^2[b(x, t)^2 p(x, t|x_0, t_0)] \tag{6.9.15}$$

i.e. Fokker-Plank equation.

6.9.2 Solving Fokker-Planck Equation

Analytical solutions of the Fokker-Planck equation can be found in the case [141]:

- when the drift vector is linear and diffusion tensor is constant (solutions are Gaussian distributions)
- if both obey some potential conditions, then solution is obtained by quadratures;
- in the sauce of one variable solution can be obtained by quadratures even if detailed balance is not valid.

In the general case obtaining the solution of the Fokker-Planck equation is difficult especially if no separation of variable possible or in the case of large numbers of parameters. The numerical methods for solving the Fokker-Planck equation can be found in [141] like simulation methods, transforming on the Schröndiger equation, numerical integration methods, analytic solutions for certain model potentials for a one variable Fokker-Planck equation, matrix continued-fraction solutions for a two variable FKPE and instationary solutions for time-varying small external fields.

The normalisation method by a suitable transformation transforms the diffusion coefficient dependent of σ_{12} to an arbitrary constant, which can be 1. This normalisation is not so suitable in the case of the low-noise limit where diffusion coefficient goes to zero. In the stationary case the probability current is constant, so that the Fokker-Planck equation becomes simpler and can be immediately integrated. The integration constants are obtained by normalisation and boundary conditions. In the case of the nonstacionary solutions, it's very hard to obtain it and it is possible only for special drift and diffusion coefficients. In the case of the vanishing drift coefficient and constant diffusion coefficient Fokker-Planck equation is called Wiener process.

The method for solving Fokker -Planck equation of the form [160]:

$$\frac{\partial P}{\partial t} = -N_{(1)}\frac{\partial P}{\partial \sigma_{12}} + N_{(2)}\frac{\partial^2 P}{\partial \sigma_{12}^2}, \tag{6.9.16}$$

where $N_{(1)}$ and $N_{(2)}$ are called advection and diffusion coefficients respectively, used by Jeremic is the method of lines [148, 83]. Also, one must concern initial conditions and the boundary condition given by:

$$\zeta(-\infty, t) = \zeta(\infty, t) = 0. \tag{6.9.17}$$

Note: *The domain in real problems is not from $-\infty$ to $+\infty$ since stresses don't have infinity range but very small finite range.*

The discretisation of previous Fokker-Planck equation (6.9.16) is done using the central difference method:

$$\frac{\partial^2 P}{\partial \sigma_{12}^2} = \frac{1}{\Delta \sigma_{12}^2}(P_{i+1} - 2P_i + P_{i-1}) \tag{6.9.18}$$

and

$$\frac{\partial P}{\partial \sigma_{12}} = \frac{1}{2\Delta\sigma_{12}}(P_{i+1} - P_{i-1}). \tag{6.9.19}$$

The central difference technique might not be so efficient in solving n-dimensional Fokker-Planck PDE. With addaptivity and reduced ordered modelling efficiently can be improved.

6.10 Smolyak Quadrature Method

Smolyak's algorithm is a method first developed to handle high dimensional quadrature [90, 91, 140, 167] and further extended to accomplish high dimensional interpolation [11]. Its basic idea is to use the solution to several low-dimensional problems in order to span the space and linearly combine these to yield the solution to the high-dimensional problem.

Usually as the final result of the process of solving stochastic system is required the expectation of some function $\Psi : \Omega \to \mathbb{R}$ i.e. $\mathbf{E}(\Psi(\theta))$. To calculate this expansion one can use quadrature formula given as

$$Q(\Psi_i) = \sum_{z=1}^{Z} w^{(z)} \Psi(\omega^{(z)}), \tag{6.10.1}$$

where $w^{(z)}$ are the weights and $\omega^{(z)}$ are the nodes.

For high dimension assume that in each dimension $\Omega_i, (i = 1, ..., m)$ is given quadrature formula $Q^{(i)}$. Assume that each of this formulas have the same number of nodes Z and integrating polynomials of degree k with the respect to the measure $dP_{\theta_i(\omega_i)}$. In that case quadrature formula $Q^{(m)}$ may be constructed as the tensor product of the onedimensional quadrature formulas:

$$Q^{(m)} = Q^{(1)} \otimes Q^{(2)} \otimes ... \otimes Q^{(m)} \tag{6.10.2}$$

i.e.

$$Q^{(m)}(\Psi) = \sum_{z_1=1}^{Z} \sum_{z_2=1}^{Z} ... \sum_{z_m=1}^{Z} w_1^{(z_1)} ... w_m^{(z_m)} \Psi(\omega_1^{(z_1)}, ..., \omega_1^{(z_1)}), \tag{6.10.3}$$

with respect to the measure $dP_\theta(\omega)$. The disadvantage of this method is that this computation need Z^m evaluations of the integrand, which is not feasible for high dimensions. In order to improve method quadrature formulas in each dimension are used such that for an $l = (l_1, ..., l_m)^t \in \mathbb{N}^m$ a quadrature formula for functions $\Psi : \Omega^{(m)} \to \mathbb{R}$ may be constructed as

$$Q_l = Q_{l_1}^{(1)} \otimes Q_{l_2}^{(2)} ... \otimes Q_{l_m}^{(m)} \tag{6.10.4}$$

which is applied to Ψ by

$$Q_{l(\Psi)} = \sum_{z_1=1}^{Z_{l_1}} ... \sum_{z_m=1}^{Z_{l_m}} w_{l_1,k_1}^{(1)} ... w_{l_m,k_m}^{(m)} \Psi(\omega_{l_1,k_1}^{(1)}, ..., \omega_{l_m,k_m}^{(m)}. \tag{6.10.5}$$

Lowest order quadrature formula uses just $Z_1 = 1$ points, so the evaluation is feasible in high dimension if $l_1 \neq 1$ for only few i.

Smolyak combined these formulas using

$$\triangle_l^{(i)} = Q_l^{(i)} - Q_{l-1}^{(i)}, \quad l \in \mathbb{N}, i = 1, ..., m, \tag{6.10.6}$$

where $Q_0^{(i)} = 0$ for all i, such that the level l formula has the form:

$$S_l^m = \sum_{l \in \mathbb{N}^m, |l| \leq m+l-1} \triangle_{l_1}^{(1)} \otimes \triangle_{l_2}^{(2)} ... \otimes \triangle_{l_m}^{(m)}, \tag{6.10.7}$$

where $|l| = l_1 + l_2 + ... + l_m$.

This formula becomes practical because combines higher order formulas in few dimensions with low order formulas in the other dimensions, so that the resulting formula is feasible.

6.11 Stochastic Collocation Method

This method was developed by Mathelin and Hussaini [114, 115] to enable application of stochastic methods to spectral discontinuous Galerkin methods and to reduce the costs of polynomial chaos method. The main idea of this method is to collocate problem in the zeros of tensor product orthogonal polynomials with the respect to the joint probability density ρ of the random variables (independent random variables). It leads to uncoupled deterministic problems, even in the case of input data which depend nonlinearly on the random variables. Also it can deal with unbounded random variables (Gaussian or exponential), as well in the case of nonindependent random variables with auxiliary density ρ_n [73, 128, 182].

The objective is to combine the strength of the two existing approaches: the high resolution of stochastic Galerkin methods resulting from polynomial approximations in random spaces, and the ease of implementation of Monte Carlo methods by sampling at discrete points in random spaces.

Babuska and his coworkers gave the theory of the collocation method for elliptic differential equation with random inputs [9]. This method is based on polynomial interpolations in the multidimensional random space.

Let us denote by $y = \{y_1, y_2, ..., y_M\}$ any point in the random space Γ, by $\prod_N$ the space of all N-variate polynomials with real coefficients, and by $\prod_N^p$ the subspace of polynomials of total degree at most p. For given collocation nodes y in the random space, interpolation of a smooth function f is obtained using polynomial $I(f)$ such that

$$I(f)(y_i) = f(y_i), \forall i = 1, ..., M. \tag{6.11.1}$$

Interpolating solution :

$$\hat{u}(y) = I(u)(y) = \sum_{k=1}^{M} u(y_k) L_k(y), \tag{6.11.2}$$

where

$$L_i(y_j) = \delta_{ij}, \quad 1 \leq i, j \leq M, \tag{6.11.3}$$

one can get procedure of solving elliptic differential equation

$$R(\hat{u}(y))|_{y_k} = 0, \forall k = 1, ..., M, \tag{6.11.4}$$

where R is residual. Thus, the stochastic collocation method is equivalent to solving M deterministic problems. Also for each k, system is naturally decoupled, and existing deterministic solvers can be readily applied. This is in contrast with stochastic Galerkin approaches where equations are in general coupled.

Like a result of numerical simulations obtained at all collocation points, the statistic can be evaluated:

$$\mathbf{E}(\hat{u})(x) = \sum_{k=1}^{M} u(y_k, x) \int_{\Gamma} L_k(y)\rho(y)dy. \tag{6.11.5}$$

6.12 Conclusion

In this chapter are reviewed numerical methods for solving stochastic problems, which one can find in the literature of stochastic plasticity [172].

The method of moment equations relies on deriving effective equations for the statistical moments. But when we speak about the nonlinear problems the moment equations are not closed and lower order moments depend on higher order moments. It is very difficult to say that method converge or not because of the hierarchy of the system. There is a possibility to truncate the moment equations, which can lead to numerical errors and even unphysical result. For that reason this method is used only in some applications [109].

The probability density method is a method of deriving the Fokker-Planck equations for the probability density function (PDF) of the random solution. The PDF is a function of the spatial-temporal variables as well as the state variables for the random solution itself. This method give us detailed statistical information of the random solution. The problem arise when it deals with the nonlinear equations. In that case this method involves the derivatives of conditional expectations. Method of PDF is a high dimensional method, transforming equation whith two dimensions to a function with five dimensions. Little bit is easier when exist approximate Fokker-Planck equation for the PDF.

The idea of Monte Carlo (MC) simulations is to sample the randomness in the SPDEs and solve the stochastic equations realization by realization. For each given realization of the randomness, the SPDEs become deterministic and can be solved by regular numerical methods. Computing the ensemble averages, solving the SPDE many times with different solutions, one can obtain the statistical information of the random solution. The Monte Carlo methods use the pseudo random number generator. They are very general and robust but very slowly.

In recent years, polynomial chaos expansion has received much attention as a promising numerical method in solving SPDEs. Using Hermite polynomials, Wiener constructed an

orthonormal random basis for expanding homogeneous chaos depending on white noise, and used it to study problems in statistical mechanics. The arising problem was that second order Wiener expansion cannot faithfully represent the dynamics of the solution. However, the inclusion of the higher order terms was technically very difficult. The Wiener original formulation is replaced with Cameron and Martin idea to use Fourier expansion. This Fourirer expansion was commonly called Wiener chaos expansion (WCE) thereafter.

Another incarnation of Wiener chaos (or polynomial chaos) expansions as a numerical method is largely due to the original work by Ghanem and Spanos [59]. They designed a new numerical method for solving elliptic equations with random coefficients. Using the Karhunen-Loève expansion (KLE), they first expand the random coefficients (a Gaussian field in space) as a series of Gaussian random variables and then the random solution represent as a Hermite expansion of those Gaussian random variables. By projecting the stochastic elliptic equations onto the probability space, they solved the equations numerically within the framework of finite element method. This method is called stochastic finite element method. Using Fourier-Hermite expansion for modelling non-Gaussian processes is also investigated [54, 56]. Ghanem's approach for solving stochastic elliptic equations was further developed and generalised by many other researchers, notably [7, 10, 89, 111, 112, 122, 158]. Xiu and Karniadakis [185] generalised the Hermite polynomial expansion to include other orthogonal polynomials, and used it to study flow-structure interactions [184, 186] and linear convection problems [77]. Zhang et al. [106, 107] combined moment perturbation method with polynomial chaos expansion, and used it to study the saturation flows in heterogeneous porous media. Hence, the source of the randomness usually contains only a small number of random variables. For those cases, the polynomial chaos expansion based on the Karhunen-Loève expansion provides powerful tools for the uncertainty quantification purpose. However, if the source randomness keeps changing and has very short correlation length in time, such as white noise, then the KLE-based method may fail to capture the randomness correctly.

Chapter 7

SFEM For Elastoplastic Body

Heterogeneities at the micro-structural level are usually subjected to a number of uncertainties. Usually it is assumed that the heterogeneous material behaves according to an elastoplastic model, but with uncertain parameters, which are modelled as random fields like Young's modulus, yield stress etc. Actually, the constitutive relations for elastoplastic body represented in a classical way in Chapter 5. now become equations with stochastic parameters. In the same way the evolution law for the internal variables becomes a stochastic law since the internal variables are also modelled as random fields.

The strategy of the propagation of uncertainties through governing differential equations can be broadly classified in two categories: the stochastic differential equation (SDE) with random forcing and SDE with random coefficient. The uncertainty associated with the coefficient term is generally attributed to the sampling/testing error in estimating the material properties or especially in geomechanics, due to inherent variability of the soil. The uncertainty in the forcing term arises when the material is subjected to dynamic loads like wind, wave or earthquake load. For the SDE with random forcing in the case of the Ito type equation [85, 141], there is a wide developed mathematics . In that case the solution is Markov process, where the probability density of the solution obeys a Fokker-Planck (FP) partial differential equation (PDE). Hence, the nonlinear SDE equation in real space Fokker-Planck equation transforms into linear deterministic PDE in probability density space. In the second case (SDE with random coefficients) a few numerical methods are proposed. The most frequently used are the perturbation method, the Monte Carlo method etc.

First attempt to propagate uncertainties through elastoplastic constitutive equations, considering Young's modulus as a random variable, was published only recently. It is done by Anders and Hori [3, 4]. They used Karhunen-Loève expansion and the polynomial chaos expansion. In the theory can be also found the probability density approach [87], which use a generic Eulerian-Lagrangian equation form of FP equation exact to second order, corresponding to any nonlinear ordinary differential equation with random coefficient and random forcing [80, 160, 161].

7.1 Fokker-Planck Equation of Elastoplasticity

The elastoplastic constitutive law is a set of linear/nonlinear ordinary differential equations (ODEs) relating the rate of the stress with the rate of the strain through the linear-nonlinear material modulus. It has the form (this can be done only in the case of smooth functions) [80, 160, 161]:

$$\frac{d\sigma_{ij}}{dt} = C^{ep}_{ijkl}\frac{d\varepsilon_{kl}}{dt}, \tag{7.1.1}$$

where constitutive tensor of forth order C^{ep}_{ijkl} can be linear or nonlinear function of σ_{ij}. The set of ODE becomes a set of stochastic differential equations (SDE) if either the material modulus or the forcing term (strain rate) becomes random [80, 160, 161].
In that way, the equation (7.1.2) becomes the equation of the shape:

$$\frac{d\sigma_{ij}(x_t,t)}{dt} = C^{ep}_{ijkl}(\sigma_{ij}, C^{e}_{ijkl}, f, U, q_*, r_*; x_t, t)\frac{d\varepsilon_{kl}}{dt}, \tag{7.1.2}$$

where C^{ep}_{ijkl} is the random, nonlinear elastoplastic coefficient tensor which is a function of a random stress tensor (σ_{ij}), random elastic moduli tensor C^{e}_{ijkl}, random yield function (f), random potential function (U), random internal variables (q_*) and random direction of internal variables (r_*). The random internal variables could be scalar (for perfectly plastic and isotropic hardening models) or second-order tensor (for translational and rotational kinematic hardening models), or fourth-order tensor or any combinations above. Denoting all random material parameters by one:

$$C_{ijkl} = [C^{e}_{ijkl}, f, U, q_*, r_*] \tag{7.1.3}$$

and introducing a random operator tensor, η_{ij}, one can write

$$\frac{d\sigma_{ij}(x,t)}{dt} = \eta_{ij}(\sigma_{ij}, C_{ijkl}, \varepsilon_{kl}; x, t) \tag{7.1.4}$$

with initial condition:

$$\sigma_{ij}(x,0) = \sigma_{ij_0}. \tag{7.1.5}$$

As it is represent in [161], equation (7.1.4) can be considered to represent a point in 9-dimensional stress $\boldsymbol{\sigma}$ space and determines the velocity for the point in that space. Actually, one has a trajectory starting out in initial point in stress space, given by initial condition, which describes the corresponding solution of the nonlinear stochastic ODE system. Imaging a cloud of initial points, described by a density $\rho(\sigma_{ij}, 0)$ in the $\boldsymbol{\sigma}$ stress

space, previous equation will describe movement of all these points. Finally one obtains Kubo's stochastic Liouville equation [98]:

$$\frac{\partial \rho(\sigma_{ij}(x,t),t)}{\partial t} = -\frac{\eta_{mn}(\sigma_{mn}(x,t), C_{mnpq}(x), \varepsilon_{pq}(x,t))\rho(\sigma_{ij}(x,t),t)}{\partial \sigma_{mn}} \tag{7.1.6}$$

where the initial condition becomes:

$$\rho(\sigma_{ij},0) = \delta(\sigma_{ij} - \sigma_{ij_0}) \tag{7.1.7}$$

and where the function δ is the Dirac delta function. Using the Van Kampen's Lemma one can obtain:

$$\langle \rho(\sigma_{ij},t)\rangle = P(\sigma_{ij},t) \tag{7.1.8}$$

or in other words the expectation of density is equal to the evolutionary probability density P of the state variable tensor σ_{ij} . The ensemble average form of previous equation was derived by Kavvas [87]:

$$\begin{aligned}\frac{\partial\langle\rho(\sigma_{ij}(x_t,t),t)\rangle}{\partial t} &= -\frac{\partial}{\partial\sigma_{mn}}[\{\langle\eta_{mn}(\sigma_{mn}(x_t,t), C_{mnrs}(x_t), \varepsilon_{rs}(x_t,t))\rangle \\ &\quad - \int_0^t d\tau \mathrm{Cov}_0[\eta_{mn}(\sigma_{mn}(x_t,t), C_{mnrs}(x_t), \varepsilon_{rs}(x_t,t)) \\ &\quad \frac{\partial\eta_{ab}(\sigma_{ab}(x_{t-\tau},t-\tau), C_{abcd}(x_{t-\tau}), \varepsilon_{rs}(x_{t-\tau},t-\tau)}{\partial\sigma_{ab}}]\}\langle\rho(\sigma_{ij}(x_t,t),t)\rangle] \\ &\quad + \frac{\partial}{\partial\sigma_{mn}}[\int_0^t d\tau \mathrm{Cov}_0\eta_{mn}(\sigma_{mn}(x_t,t), C_{mnrs}(x_t), \varepsilon_{rs}(x_t,t)); \\ &\quad \eta_{ab}(x_{t-\tau},t-\tau), C_{abcd}(x_{t-\tau}, \varepsilon_{cd}(x_{t-\tau},t-\tau))]\}\frac{\partial\langle\rho(\sigma_{ij}(x_t,t),t)\rangle}{\partial\sigma_{ab}}],\end{aligned} \tag{7.1.9}$$

where covariance Cov_0 is the time ordered covariance function defined by:

$$\begin{aligned}\mathrm{Cov}_0[\eta_{mn}(x,t_1), \eta_{ab}(x,t_2)] &= \langle\eta_{mn}(x,t_1)\eta_{ab}(x,t_2)\rangle \\ &\quad - \langle\eta_{mn}(x,t_1)\rangle\langle\eta_{ab}(x,t_2)\rangle.\end{aligned} \tag{7.1.10}$$

Kavvas derived a generic Eulerian-Lagrangian form of FPK equation:

$$\begin{aligned}\frac{\partial P(\sigma_{ij}(x_t,t),t)}{\partial t} &= \frac{\partial}{\partial\sigma_{mn}}[\{\langle\eta_{mn}(\sigma_{mn}(x_t,t), C_{mnrs}(x_t), \varepsilon_{rs}(x_t,t))\rangle + \\ &\quad \int_0^t d\tau \mathrm{Cov}_0[\frac{\partial\eta_{mn}(\sigma_{mn}(x_t,t), C_{mnrs}(x_t), \varepsilon_{rs}(x_t,t))}{\partial\sigma_{ab}}; \\ &\quad \eta_{ab}(\sigma_{ab}(x_{t-\tau},t-\tau), C_{abcd}(x_{t-\tau}), \varepsilon_{cd}(x_{t-\tau},t-\tau)]\}P(\sigma_{ij}(x_t,t),t)] \\ &\quad + \frac{\partial^2}{\partial\sigma_{mn}\partial\sigma_{ab}}[\{\int_0^t d\tau \mathrm{Cov}_0[\eta_{mn}(\sigma_{mn}(x_t,t), C_{mnrs}(x_t), \varepsilon_{rs}(x_t,t)); \\ &\quad \eta_{ab}(\sigma_{ab}(x_{t-\tau},t-\tau), C_{abcd}(x_{t-\tau}), \varepsilon_{cd}(x_{t-\tau},t-\tau))]\}P(\sigma_{ij}(x_t,t),t)].\end{aligned} \tag{7.1.11}$$

The final result of FPKE is that equation (7.1.11) is linear in terms of its unknown (the probability density $P(\sigma_{ij}, t)$ of the state variable tensor σ_{ij}), while the original equation is nonlinear. In this equation the space location $x_{t-\tau}$ is unknown while x_t is known. To get unknown space location one assumes the small strain theory and use the strain rate:

$$d\boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}}\tau = \frac{x_t - x_{t-\tau}}{x_t}, \tag{7.1.12}$$

where one finds:

$$x_{t-\tau} = (1 - \dot{\boldsymbol{\varepsilon}}\tau)x_t. \tag{7.1.13}$$

Once the probability density function $P(\boldsymbol{\sigma}, t)$ is obtained it can be used to obtain the mean of the state variable ($\boldsymbol{\sigma}$) by usual expectation operation

$$\langle \sigma(t) \rangle = \int \boldsymbol{\sigma}(t) P(\boldsymbol{\sigma}(t)) d\boldsymbol{\sigma}(t). \tag{7.1.14}$$

As it is shown in [161, 80, 160], the mean stress tensor (σ_{ij}) is given as:

$$\begin{aligned} \frac{\langle d\sigma_{ij}(x,t) \rangle}{dt} &= \langle \eta_{ij}(\sigma_{ij}(x_t,t), C_{ijkl}(x_t), \varepsilon_{kl}(x_t,t)) \rangle \\ &+ \int_0^t d\tau \mathrm{Cov}_0[\frac{\partial \eta_{ij}(\sigma_{ij}(x_t,t), C_{ijkl}(x_t), \varepsilon_{kl}(x_t,t)}{\partial \sigma_{ab}}; \\ &\eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), C_{abcd}, \varepsilon_{cd}(x_{t-\tau}, t-\tau))]. \end{aligned} \tag{7.1.15}$$

The difficulty is that the stress tensor appearing within $\eta_{ij}(\cdot)$ in the covariance term is random and needs to be treated accordingly. To solve this one can use the perturbation method with respect to the mean or by solving FPKE.

7.1.1 Probabilistic behaviour of the Drucker Prager Model

In the section 5.6.5. is described the Drucker-Prager material model whose yield criteria is given by equation (5.6.10). The stiffness tensor is given by [80]:

$$D^{ep}_{ijkl} = \begin{cases} 2G\delta_{ik}\delta_{jl} + (K - \frac{2}{3}G)\delta_{ij}\delta_{kl} & , \phi < 0 \vee (\phi = 0 \wedge d\phi < 0) \\ 2G\delta_{ik}\delta_{jl} + (K - \frac{2}{3}G)\delta_{ij}\delta_{kl} - \frac{A_{ij}A_{kl}}{B + K_p} & , \phi = 0 \vee d\phi = 0 \end{cases} \tag{7.1.16}$$

where tensor $A_{kl} = D_{pqkl}\frac{\partial \phi}{\partial \sigma_{pq}}$ and scalars $B = \frac{\partial \phi}{\partial \sigma_{rs}} D_{rstu} \frac{\partial \phi}{\partial \sigma_{tu}}$ and $K_P = -\frac{\partial \phi}{\partial q_n} r_n$ are defined by equations in [80].

In this case the function η is given by:

$$\eta = \begin{cases} 2G\frac{d\varepsilon_{12}}{dt} & \phi < 0 \vee (\phi = 0 \wedge d\phi < 0) \\ (2G - \frac{4G^2(\frac{\partial \phi}{\partial \sqrt{J_2}} \frac{\sqrt{J_2}}{\partial \sigma_{12}})^2}{B + K_P})\frac{d\varepsilon_{12}}{dt} & , \quad \phi = 0 \vee d\phi = 0. \end{cases} \tag{7.1.17}$$

From this equation one obtains two Fokker-Plank equations: for elastic region (pre-yield) and for plastic region (post yield). In the case of elastic region one has:

$$\begin{aligned}\frac{\partial P(\sigma_{12},t)}{\partial t} =& -\frac{\partial}{\partial \sigma_{12}}[\langle 2G\frac{d\varepsilon_{12}}{dt}\rangle P(\sigma_{12},t)] \\ &+\frac{\partial^2}{\partial \sigma_{12}^2}[\{\int_0^t d\tau \mathrm{Cov}_0[2G\frac{d\varepsilon_{12}}{dt}(t); \\ &2G\frac{d\varepsilon_{12}}{dt}(t-\tau)]\}P(\sigma_{12},t)],\end{aligned} \tag{7.1.18}$$

while for elastoplastic region using some assumptions [80] one gets:

$$\begin{aligned}\frac{\langle d\sigma_{12}(t)\rangle}{dt} =& \langle G^{ep}(t)\frac{d\varepsilon_{12}}{dt}(t)\rangle \\ &+\int_0^t d\tau \mathrm{Cov}_0[\frac{\partial}{\partial \sigma_{12}}(G^{ep}(t)\frac{d\varepsilon_{12}}{dt}(t); \\ &G^{ep}(t-\tau)\frac{d\varepsilon_{12}}{dt}(t-\tau)],\end{aligned} \tag{7.1.19}$$

where G^{ep} is given by:

$$G^{ep} = (2G - \frac{4G^2(\frac{\partial \phi}{\partial \sqrt{J_2}}\frac{\sqrt{J_2}}{\partial \sigma_{12}})^2}{B+K_P}) \tag{7.1.20}$$

$$G^{ep} = 2G - \frac{G^2}{G+9K\alpha^2+\frac{1}{\sqrt{3}}I_1\alpha'}. \tag{7.1.21}$$

All derivatives in G^{ep} are for the constant values of σ_{12} .They are given in the deterministic sense, while the derivatives appearing in the previous equation are random differentiations and need to be treated accordingly.

7.1.2 Probabilistic behaviour of CAM-CLAY Model

In the case of the CAM-CLAY model the yield function is given by equation (5.6.11). The function η can be obtained in the form:

$$\eta_{ij} = \begin{cases}[2G\delta_{ik}\delta_{jl} + (K-\frac{2}{3}G)\delta_{ij}\delta_{kl}]\frac{d\varepsilon_{kl}}{dt} & ,\phi<0 \vee (\phi=0 \wedge d\phi<0 \\ [2G\delta_{ik}\delta_{jl} + (K-\frac{2}{3}G)\delta_{ij}\delta_{kl} - \frac{A_{ij}A_{kl}}{B+K_p}]\frac{d\varepsilon_{kl}}{dt} & ,\phi=0 \vee d\phi=0\end{cases} \tag{7.1.22}$$

Similar as for the Drucker-Prager model, one can obtain the FKPE:

$$\begin{aligned}\frac{\partial P(\sigma_{12},t)}{\partial t} &= -\frac{\partial}{\partial \sigma_{12}}\{P(\sigma_{12},t)N_{(1)}\} + \frac{\partial^2}{\partial \sigma_{12}^2}\{P(\sigma_{12},t)N_{(2)}\} \\ &= -\frac{\partial}{\partial \sigma_{12}}[P(\sigma_{12},t)N_{(1)}\} - \frac{\partial}{\partial \sigma_{12}}\{P(\sigma_{12},t)N_{(2)}\}] \\ &= -\frac{\partial \xi}{\partial \sigma_{12}},\end{aligned} \tag{7.1.23}$$

where $N_{(1)}$ and $N_{(2)}$ are coefficients of the FKPE and they are called advection and diffusion coefficients.

The initial conditions are separated on the initial conditions for the pre-yield and for the post-yield case. The initial condition for the pre-yield solution of the PDF is deterministic and means that in the initial time all mass is concentrated at $\sigma_{12} = 0$ i.e. matched with the Dirac delta function :

$$P(\sigma_{12}, 0) = \delta(\sigma_{12}). \tag{7.1.24}$$

For the post-yield case there will be some distribution of σ_{12} corresponding to the solution of the pre-yield probabilistic behaviour to begin with. The probability mass within the system is conserved i.e. mathematically speaking :

$$\xi(\sigma_{12}, t)|_{boundery} = 0. \tag{7.1.25}$$

The domain for the stress in a theory is from $-\infty$ to $+\infty$, so that boundary conditions are given as:

$$\xi(-\infty, t) = \xi(\infty, t) = 0 \tag{7.1.26}$$

under the assumption that the parameters have normal distribution. In the case of the exponential distribution the domain is from 0 to $+\infty$.

7.2 SFEM using Series Expansion

The series expansion approach to nonlinear elastoplastic problem is based on the generalisation of the stochastic finite element method for linear elastic bodies developed by Ghanem and Spanos [59], who applied the Karhunen-Loève expansion and the polynomial chaos expansion for stochastic material properties and fields variables respectively. Anders and Hori [3] proposed theory of bounding bodies providing upper and lower bounds for the mean of field variables. The two bounding bodies are rigorously obtained from a given distribution of material properties. In particular, the nonlinear constitutive equations are most difficult to deal with, since they contain triple products of stochastic variables. In order to approximatly satisfy the constitutive equations one can use two fictitious bounding bodies.

7.2.1 Theory of Bounding Bodies

The concept of bounding bodies originally comes from micromechanics. It evaluates upper and lower bounds for effective properties of materials with stochastically varying microstructures [126, 163]. The bounding media are two fictitious media for stochastically varying bodies, and their responses provide optimistic and pessimistic estimates on the mean behaviour in a sense that the total elastic strain energy stored in the bounding media bounds the mean of total elastic strain energy.

Observing the elastic perfectly plastic material one can define bounding bodies with stochastically varying elasticity tensor C_{ijkl} and a deterministic (but non-homogeneous) yield function f and assume associated flow rule. As it is shown in [3], a variational problem, for the rate of traction prescribed on the boundary, is formulated by the functional:

$$J(\dot{\vec{\boldsymbol{u}}}, \dot{\gamma}, \vec{C}) = -\int_V \frac{1}{2}(\dot{u}_{i,j} - \dot{\gamma}(\bigtriangledown f)_{ij}) C_{ijkl}(\dot{u}_{k,l} - \dot{\gamma}(\bigtriangledown f)_{kl}) dV + \int_{\partial V} \dot{u}_i \dot{T}_i dS, \tag{7.2.1}$$

where $\dot{T}_i$ is the rate of traction prescribed on the boundary, $\dot{\boldsymbol{u}}$ is the displacement rate and $\dot{\gamma}$ is the plastic strain rate multiplier. Also, one can obtain a variational problem for the stress rate $\dot{\sigma}_{ij}$ as the functional:

$$I(\dot{\vec{\boldsymbol{\sigma}}}, \dot{\gamma}, \vec{C}) = \int_V (\frac{1}{2}\dot{\sigma}_{ij} C^{-1}_{ijkl} \dot{\sigma}_{kl} + \dot{\gamma}(\bigtriangledown f)_{ij}\dot{\sigma}_{ij} + \mu_j \dot{\sigma}_{ij,j}) dV. \tag{7.2.2}$$

J and I attain the maximum and minimum that coincide with:

$$\ddot{U}^e = \int_V \frac{1}{2}\dot{\varepsilon}^e_{ij} C_{ijkl} \dot{\varepsilon}^e_{kl} dV, \tag{7.2.3}$$

so that one has following inequality:

$$J(\dot{\vec{\boldsymbol{u}}}, \langle \vec{\boldsymbol{C}} \rangle) < \langle \ddot{U}^e \rangle < I(\dot{\vec{\boldsymbol{\sigma}}}, \langle \vec{\boldsymbol{C}}^{-1} \rangle^{-1}). \tag{7.2.4}$$

Using previous inequality one can define two bounding bodies V^+ and V^-, with $C^+_{ijkl} = \langle C^{-1}_{ijkl} \rangle^{-1}$ or $C^{-1}_{ijkl} = \langle C_{ijkl} \rangle$, respectively, for the present nonlinear case.

Since we have the stochastic inherent in the material properties, the field variables $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{\varepsilon}}$ and $\vec{\boldsymbol{\sigma}}$ as well as $\vec{\boldsymbol{C}}^{ep}$ will also become stochastic (see Chapter 5.). The main goal of equations in Chapter 5. is to provide a relation between probability distribution of material properties and unknown joint distribution of these field variables. The major difficulty in finding the joint distribution is the nonlinear dependence of the elastoplastic constitutive tensor and the stress tensor.

Starting from equations in Chapter 5. and the probability density of Young's modulus $P(E)$, one can derive the joint probability density $P(E, \vec{\boldsymbol{u}}, \vec{\boldsymbol{\varepsilon}}, \vec{\boldsymbol{\sigma}})$. The major difficulty in solving the above problem is the nonlinear coupling of stochastic field variables. To overcome this difficulty, Anders and Hori [3, 4] solved problem taking the perturbation expansion at the stochastic mean behaviour. That is, the yield function ϕ is expanded as:

$$\phi(\vec{\boldsymbol{\sigma}}, \gamma) = \phi|_{\langle\rangle} + (\partial_{\vec{\boldsymbol{\sigma}}}\phi)|_{\langle\rangle}) : (\vec{\boldsymbol{\sigma}} - \langle \vec{\boldsymbol{\sigma}} \rangle) + \partial_\gamma \phi|_{\langle\rangle}(\gamma - \langle\gamma\rangle) + ... \tag{7.2.5}$$

and only first term is considered. The accuracy of the first approximation for the ideally plastic case has been shown in [3] to be high even for the large value of the covariance of the

Young's modulus. Having only Young's modulus E as stochastic parameter the stochastic elasticity tensor can be expressed as:

$$C_{ijkl}(x,\omega) = E(x,\omega)C^*_{ijkl}(x), \tag{7.2.6}$$

where C_{ijkl} is expressed in terms of deterministic Poisson's ratio ν as:

$$C^*_{ijkl} = \frac{\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{1}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{7.2.7}$$

The substitution of the constitutive tensor in equation for the stress rate leads to:

$$\dot{\vec{\boldsymbol{\sigma}}}(\omega) = \vec{\boldsymbol{C}}(\omega) : (\dot{\vec{\boldsymbol{\varepsilon}}}_t(\omega) - \dot{\gamma}(\omega)\vec{\nabla}\phi(\vec{\boldsymbol{\sigma}}_t(\omega))) \tag{7.2.8}$$

$$\phi(\vec{\boldsymbol{\sigma}}_t(\omega)) \leq 0. \tag{7.2.9}$$

The stochastic stress rate approaches the exact one as the variance of E goes to zero and $V^{\pm}$ become close to each other. In the equation (7.3.8) we can notice triple product of random field variables and a function ϕ, which is a nonlinear function of $\vec{\boldsymbol{\sigma}}_t$. $V^{\pm}$ can bound the stochastic equation of field variables providing the evaluation of the gradients on a suitable stress field. Hence, the computation of the mean stress is very tedious and require the triple product of the random variables.

The theory of bounding bodies is based on the next assumptions:

- the stress in bounding bodies will provide bounds for the mean,
- the stress in the bounding bodies is related to the mean of the strain and plastic coefficient.

Since the field variables in $V^{\pm}$ approximately evaluate the mean of stochastic field variables, we can assume $\dot{\vec{\boldsymbol{\varepsilon}}}^{\pm} = \langle\dot{\vec{\boldsymbol{\varepsilon}}}\rangle$ and $\gamma^{\pm} = \langle\gamma\rangle$. Now, for the bounding bodies $V^{\pm}$ we can obtain expressions based on first assumption:

$$\dot{\vec{\boldsymbol{\sigma}}}(\omega)^{+} = \vec{\boldsymbol{C}}(\omega) : (\dot{\vec{\boldsymbol{\varepsilon}}}_t(\omega) - \dot{\gamma}(\omega)\vec{\nabla}\phi(\vec{\boldsymbol{\sigma}}^{+}_t(\omega))) \tag{7.2.10}$$

$$\phi(\vec{\boldsymbol{\sigma}}^{+}_t(\omega)) \leq 0 \tag{7.2.11}$$

and

$$\dot{\vec{\boldsymbol{\sigma}}}^{-} = \vec{\boldsymbol{C}}(\omega) : (\dot{\vec{\boldsymbol{\varepsilon}}}_t(\omega) - \dot{\gamma}(\omega)\vec{\nabla}\phi(\vec{\boldsymbol{\sigma}}^{-}_t(\omega))) \tag{7.2.12}$$

$$\phi(\vec{\boldsymbol{\sigma}}^{-}_t(\omega)) \leq 0. \tag{7.2.13}$$

The second assumption leads us to:

$$\dot{\vec{\boldsymbol{\sigma}}}^{+} = \vec{\boldsymbol{C}}^{+} : (\langle\dot{\vec{\boldsymbol{\varepsilon}}}_t\rangle - \langle\dot{\gamma}\rangle\vec{\nabla}\phi(\vec{\boldsymbol{\sigma}}^{+}_t)) \tag{7.2.14}$$

$$\dot{\vec{\boldsymbol{\sigma}}}^{-} = \vec{\boldsymbol{C}}^{-} : (\langle\dot{\vec{\boldsymbol{\varepsilon}}}_t\rangle - \langle\dot{\gamma}\rangle\vec{\nabla}\phi(\vec{\boldsymbol{\sigma}}^{-}_t)). \tag{7.2.15}$$

These equations represent the bounding body approximation, which enables much easier solving of the nonlinear problem. The nonlinearity is removed, as well as triple products. The equations are exact in the deterministic limit i.e. in the limit as the variance of E approaches zero.

Using the bounding body approximation, we can relax the governing equations for stochastic displacement rate, as though the nonlinear stochastic dependence of $\boldsymbol{C}^{ep}$ on $\vec{\boldsymbol{\sigma}}$ and γ has been excluded. That is, $\vec{\boldsymbol{C}}^{ep}$ is approximately evaluated as:

$$\vec{\boldsymbol{C}}^{ep} = \begin{cases} \vec{E}(\vec{\boldsymbol{C}}^* - [(\vec{\boldsymbol{C}}^* : \partial_{\vec{\boldsymbol{\sigma}}}\phi) \otimes (\partial_{\vec{\boldsymbol{\sigma}}}\phi : \vec{\boldsymbol{C}}^*)]/[\partial_{\vec{\boldsymbol{\sigma}}}\phi : \vec{\boldsymbol{C}}^* : \partial_{\vec{\boldsymbol{\sigma}}}\phi - \partial_\gamma\phi|_E]) \\ E\vec{\boldsymbol{C}}^* \end{cases} \tag{7.2.16}$$

where $\vec{\boldsymbol{C}}^*$ is given as $\vec{\boldsymbol{C}}$ / E. Since the quantities in the large parenthesis are evaluated at $V^\pm$, we evaluate $\partial_{\vec{\boldsymbol{\sigma}}}\phi, \partial_\gamma\phi, \phi$, and $\dot{\phi}$ at $(\vec{\boldsymbol{\sigma}}^\pm; \gamma^\pm)$ and approximate $\partial_\gamma\phi|_E$ as $\partial_\gamma\phi|_{E^\pm}$.

Enforcing bounding body approximation one uses the stress $\vec{\boldsymbol{\sigma}}^\pm$ in fictitious $V^\pm$ which must satisfy the equilibrium and the yield conditions. The stress $\vec{\boldsymbol{\sigma}}^\pm$ satisfies the equilibrium since approximates the mean stress automatically satisfiying $\nabla \cdot \langle \vec{\boldsymbol{\sigma}}_t \rangle = 0$ and coincides with it in the deterministic limit.

The elastoplastic constitutive tensor then is represent as the product of stochastic Young's modulus and nonlinear part:

$$\vec{\boldsymbol{C}}_t^{ep}(\omega) = E(\omega)\vec{\Lambda}(\vec{\boldsymbol{C}}^*, \phi(\vec{\boldsymbol{\sigma}}_t(\omega)), \tag{7.2.17}$$

or in bounding body approximation as:

$$\vec{\boldsymbol{C}}_t^{ep}(\omega) = E(\omega)\vec{\Lambda}(\vec{\boldsymbol{C}}^*, \phi(\vec{\boldsymbol{\sigma}}_t^+(\omega)). \tag{7.2.18}$$

7.2.2 SFEM with Bounding Body Approximation

As it is pointed out in the previous chapter, the series expansion give us a stochastic matrix equation. The governing equation for the displacement increment then would be:

$$[^{j-1}\mathbf{K}_i](E(\omega), [^{j-1}\boldsymbol{\sigma}_i](\omega))[^j\triangle \boldsymbol{U}_i(\omega) = [\boldsymbol{F}_i^{ext}] - [^{j-1}\boldsymbol{F}_i^{int}](\omega) \tag{7.2.19}$$

where i indicate the i-th load step and j inidicate the j-th iteration during one load step. On the right hand side of equation first term is deterministic external force and the second is stochastic internal force for unbalanced step [3, 4]. The unknown is $[^j\triangle \boldsymbol{U}_i(\omega)]$ or a stochastic displacement increment.

The appliance of the theory of the bounding bodies on the equation (7.3.19) give us $[^{j-1}\boldsymbol{\sigma}_i^\pm]$ instead of $[^{j-1}\boldsymbol{\sigma}_i]$ or in other words the stress on the bounding bodies $V^\pm$. The stochastic element stiffness matrix is assembled from the stochastic element stiffness matrices, while the elastoplastic constitutive tensor is given by the product of the stochastic and deterministic part on the bounding body. Although, the approximation with bounding stress is not necessary because of possiblity to use the series expansion of the stress.

According to the backword implicit Euler algorithm and the theory of bounding bodies, one finally gets:

$$[^j\boldsymbol{\sigma}_i^+] - [^{j-1}\boldsymbol{\sigma}_i^+] = [\boldsymbol{C}^+](\langle[^j\boldsymbol{\varepsilon}_i] > - < [^{j-1}\boldsymbol{\varepsilon}_i]\rangle - \langle {}^j\triangle\gamma_i\rangle[\bigtriangledown\phi]([^j\boldsymbol{\sigma}_i^+]), \tag{7.2.20}$$

$$\phi([^j\boldsymbol{\sigma}_i^+]) = 0, \tag{7.2.21}$$

where the unknown are $[^j\boldsymbol{\sigma}_i^+]$ and ${}^j\triangle\gamma_i$.

The computation of the solution is furnished using the KL and PC expansions and the Galerkin method for solving a system of linear equations for the unknown coefficients of the PC expansion. Update of coefficient is trivial, so that the strain increment update, as well as the stress increment update can be obtained. The process of controlling the iteration is obtained using the convergence of the right hand side of equation. When V^+ is used, the Newton-Raphson iterations convergence is significantly increased since the convergence of the stochastic displacement and the deterministic stress in V^+ is controlled by the force norm criteria. The criteria is given for the mean stress and is equal to the tangential elastoplastic matrix corresponding to V^+. When the PC expansion is used, the variance estimate obtained from the truncated expansion is an increasing function of the number of terms used. This expansion always gives an underestimate of the unknown quantity. Although, this effect is cancelled with the upper bounding body where variability of response provide an overestimate of the variance.

7.3 Probabilistic FEM using Direct Partial Differential Method

Ning, Wilson and Jiashou [127] gave a solution of probabilistic finite element methods for elastic-plastic materials of 3-D case by using the direct partial differential method. Their approach is based on incremental theory of plasticity according to modified initial stress method. The modified initial stress method is proposed by Zhuo [127] and it incorporates advantages of variational stiffness method as well as the constant stiffness method. The process of iterative computation of gradients is performed simultaneously with the computation of mean values of displacement and stress. This approach is based on computing partial differentials with respect to the random variables. This can cause small efficiency. To solve this problem authors used adjoint vector method.

Authors observed random 3-D problem discribed with random elastic modulus E, the Poisson ratio ν and strength parameters including the coefficient of internal friction f and cohesive force c. As the yield criteria they used the Mohr-Coulomb criteria.

The elastoplastic FEM equilibrium equation can be written as:

$$(\boldsymbol{K}_e - \boldsymbol{K}_p)(\Delta\boldsymbol{u}^e - \Delta\boldsymbol{u}^{'}) = \Delta\boldsymbol{P}, \tag{7.3.1}$$

where $\Delta\boldsymbol{P}$ is the load increment, $\Delta\boldsymbol{u}^e, \Delta\boldsymbol{u}^{'}$ are the elastic displacement increment and the additional displacement increment caused by plastic effect and $\boldsymbol{K}_e, \boldsymbol{K}_p$ are elastic and

plastic stiffness matrices respectively. Rewriting above equation in a set of sub-incremental and iterative equations and than applying differentiation method for the i -th iteration of the k-th loading step, equation will become:

$$\boldsymbol{K}_e \frac{\partial \Delta\Delta \boldsymbol{u}_i}{\partial X} = \frac{\partial \Delta\Delta \boldsymbol{P}_i}{\partial X} - \frac{\partial \Delta \boldsymbol{K}_e}{\partial X} \Delta\Delta \boldsymbol{u}_i, \tag{7.3.2}$$

$$\frac{\partial \Delta\Delta \boldsymbol{P}_i}{\partial X} = \frac{\partial \Delta \boldsymbol{P}_i}{\partial X} - \frac{\partial \Delta \boldsymbol{P}_{i-1}}{\partial X} = \sum_e \boldsymbol{B}^T \left(\frac{\partial \Delta \boldsymbol{\sigma}^p_{i-1}}{\partial X} - \frac{\partial \Delta \boldsymbol{\sigma}^p_{i-2}}{\partial X}\right) dv, \tag{7.3.3}$$

$$\frac{\partial \Delta \boldsymbol{u}_i}{\partial X} = \frac{\partial \Delta \boldsymbol{u}_{i-1}}{\partial X} + \frac{\partial \Delta\Delta \boldsymbol{u}_i}{\partial X}, \tag{7.3.4}$$

$$\frac{\partial \Delta \boldsymbol{\sigma}_i}{\partial X} = \frac{\partial \Delta (\boldsymbol{D}_{ep})_i}{\partial X} B \Delta u_i + (\boldsymbol{D}_{ep})_i B \frac{\partial \Delta \boldsymbol{u}_i}{\partial X}, \tag{7.3.5}$$

$$\frac{\partial \Delta \boldsymbol{\sigma}^p}{\partial X} = \frac{\partial \boldsymbol{D}_p}{\partial X} \boldsymbol{B} \Delta \boldsymbol{u} + \boldsymbol{D}_p B \frac{\partial \Delta \boldsymbol{u}}{\partial X}, \tag{7.3.6}$$

where $\boldsymbol{D}_{ep}$ is the elastoplastic stress-strain matrix defined with

$$d\boldsymbol{\sigma} = (\boldsymbol{D}_e - \boldsymbol{D}_p) d\boldsymbol{\varepsilon} = \boldsymbol{D}_{ep} d\boldsymbol{\varepsilon} \tag{7.3.7}$$

and X is one of presented random variables. $(\boldsymbol{D}_{ep})_i$ is a function of stress $\Delta\boldsymbol{\sigma}_{i-1}, E, \nu$ as well as strength parameters and it has different forms depending of the type of a random variable X. With $\Delta\Delta\boldsymbol{u}$ is denoted the cotumn matrix of the sub-incremental displacement, $\boldsymbol{P}_0$ is the residual unbalanced force of the previous loading step, and $\Delta\Delta\boldsymbol{P}$ is the unbalanced force of the present loading step.

Within each iteration one has to perform the forward and back substitution of the global stiffness matrix. This is solved by using the adjoint method. The adjoint vector method is used for the calculation of differentiation of incremental stress and displacement for plastic elements. On this method is spent almost all computational time during the step.

In calculation they show that the random material friction coefficient has the most significant influence on the standard deviation of displacement. The influences of both, elastic and Poisson's ratio, are larger than in the case of linear elastic material. Also, they observed random variables, not random fields. Although, in the case of random fields the theory remains the same just involving more random variables toward random field discretisation.

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7.4 Conclusion

Anders and Hori were first who propagate randomness through the elastoplastic constitutive equations using the bounding media theory and the perturbation method. They used

the Taylor expansion, which limits this approach on the problems with small variation. Another disadvantage of this method, called "closure-problem", appears during calculation of the lower-order moments. The calculation of lower-order moments is impossible without the information of the higher-order moments.

The Monte-Carlo method is also used in order to propagate the uncertainties in the elastoplastic problem. The disadvantage of this method is a large computational effort and typically requirement of 50 000 or more realizations per random variable in order to satisfy statistical consistency. Therefore, this method becomes impractical.

The Fokker-Planck method on other hand predicate the mean behaviour exactly but it slightly over predicate the standard deviation. The main reason for that are the Dirac delta initial conditions. The numerical solution approximated with the Gaussian distribution of mean zero and standard deviation has the error about 10^{-5}. To minimize this error it is necessary to do better approximation of the Dirac initial condition on the expense of computational cost.

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